Metric Diophantine approximations with a fixed matrix

Vasiliy Neckrasov

Brandeis University

June 5, 2025

Introduction

Notation

We will use the following conventions:

- m and n are positive integers and $d = m + n$. $\mathbf{M}_{n,m}$ is the set of all real $n \times m$ matrices.

Notation

We will use the following conventions:

- m and n are positive integers and $d = m + n$. $\mathbf{M}_{n,m}$ is the set of all real $n \times m$ matrices.
-
$$\Theta = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,m} \\ \cdots & \cdots & \cdots \\ \theta_{n,1} & \cdots & \theta_{n,m} \end{pmatrix} \in \mathbf{M}_{n,m}, \quad \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \in \mathbb{R}^n$$

  — a pair of a real $n \times m$ matrix and n-vector.

Notation

We will use the following conventions:

- m and n are positive integers and $d = m + n$. $\mathbf{M}_{n,m}$ is the set of all real $n \times m$ matrices.
-
$$\Theta = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,m} \\ \cdots & \cdots & \cdots \\ \theta_{n,1} & \cdots & \theta_{n,m} \end{pmatrix} \in \mathbf{M}_{n,m}, \quad \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \in \mathbb{R}^n$$

  — a pair of a real $n \times m$ matrix and n-vector.
- $|\cdot|$ is the supremum norm of a vector, and $||\cdot||_{\mathbb{Z}^n}$ is the distance to the nearest integer vector function.

Notation

- Classical homogeneous Diophantine approximations study the form

$$||\Theta\mathbf{q}||_{\mathbb{Z}^n}$$

- In the inhomogeneous setting, we replace the linear for with affine: we study

$$||\Theta\mathbf{q}-\boldsymbol{\eta}||_{\mathbb{Z}^n}$$

More specifically, we want to compare the values of this forms with values of certain functions:

Uniform and asymptotic approximations

Fix a non-increasing real function g. We study the system of inequalities

$$\begin{cases} ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n \leq g(T) \\ |\mathbf{q}|^m \leq T \end{cases} \qquad (1)$$

Uniform and asymptotic approximations

Fix a non-increasing real function g. We study the system of inequalities

$$\begin{cases} ||\Theta \mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n \leq g(T) \\ |\mathbf{q}|^m \leq T \end{cases} \tag{1}$$

In both homogeneous and inhomogeneous setups, two types of questions are of interest:

- Uniform approximations: does the system (1) have an integer solution $\mathbf{q}$ for all T large enough? $\widehat{\mathbf{D}}_{n,m}[g]$
- Asymptotic approximations: does the system (1) have an integer solution $\mathbf{q}$ for an unbounded set of T? $\widehat{\mathbf{W}}_{n,m}[g]$

Main definitions

Definition (g-Dirichlet pairs)

By $\widehat{\mathbf{D}}_{n,m}[g]$ we will denote the set of pairs $(\Theta, \boldsymbol{\eta})$ for which the system

$$\begin{cases} ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n \leq g(T) \\ |\mathbf{q}|^m \leq T \end{cases}$$

has an integer solution $\mathbf{q}$ for all T large enough. *Homogeneous case:* $\mathbf{D}_{n,m}[g]$.

Main definitions

Definition (g-Dirichlet pairs)

By $\widehat{\mathbf{D}}_{n,m}[g]$ we will denote the set of pairs $(\Theta, \boldsymbol{\eta})$ for which the system

$$\begin{cases} ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n \leq g(T) \\ |\mathbf{q}|^m \leq T \end{cases}$$

has an integer solution $\mathbf{q}$ for all T large enough. *Homogeneous case:* $\mathbf{D}_{n,m}[g]$.

Definition (g-approximable pairs)

By $\widehat{\mathbf{W}}_{n,m}[g]$ we will denote the set of pairs $(\Theta, \boldsymbol{\eta})$ for which the inequality

$$||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n \leq g(|\mathbf{q}|^m)$$

has infinitely many integer solutions $\mathbf{q}$. *Homogeneous case:* $\mathbf{W}_{n,m}[g]$.

Sections of sets of pairs

In the inhomogeneous case we can study three types of objects:

- Sets of pairs $(\Theta, \boldsymbol{\eta})$, satisfying some restrictions: $\widehat{\mathbf{D}}_{n,m}[g]$, $\widehat{\mathbf{W}}_{n,m}[g]$.
- Sets of matrices Θ for a fixed "shift vector" $\boldsymbol{\eta}$ ("sections with a fixed vector"): $\widehat{\mathbf{D}}^{\boldsymbol{\eta}}_{n,m}[g]$, $\widehat{\mathbf{W}}^{\boldsymbol{\eta}}_{n,m}[g]$.
- Sets of vectors $\boldsymbol{\eta}$ for a fixed matrix Θ ("sections with a fixed matrix"): $\widehat{\mathbf{D}}^{\Theta}_{n,m}[g]$, $\widehat{\mathbf{W}}^{\Theta}_{n,m}[g]$.

First two types: rather well studied.

Third type: not so many results, heavily depends on Θ.

Overview of results

Homogeneous uniform approximation

We will use the notation: $f_a(T) = \frac{1}{T^a}$, in particular, $f_1(T) = \frac{1}{T}$.

Theorem (Dirichlet)

Any matrix $\Theta \in \mathbf{M}_{n,m}$ is f_1-Dirichlet:

$$\mathbf{D}_{n,m}\left[f_1\right] = \mathbf{M}_{n,m}.$$

Theorem (Dirichlet improvable matrices have measure zero)

If $\varepsilon > 0$ is small enough, the set of ε-Dirichlet improvable matrices has Lebesgue measure zero:

$$m\left(\mathbf{D}_{n,m}\left[\varepsilon f_1\right]\right) = 0.$$

Homogeneous asymptotic approximation

Theorem (Dirichlet)

Any matrix $\Theta \in \mathbf{M}_{n,m}$ is f_1-approximable:

$$\mathbf{W}_{n,m}[f_1] = \mathbf{M}_{n,m}.$$

Theorem (Khintchine-Groshev)

The set $\mathbf{W}_{n,m}[f]$ has zero measure if the series

$$\sum_{k=1}^{\infty} f(k)$$

converges, and full measure if it diverges.

Badly approximable matrices

Badly approximable matrices are those for which the asymptotic Dirichlet's theorem cannot be improved more than by a multiplicative constant:

$$\mathbf{BA}_{n,m} = \{\Theta \in \mathbf{M}_{n,m} : \ ||\Theta \mathbf{q}||_{\mathbb{Z}^n}^n > \frac{c}{|\mathbf{q}|^m} \text{ for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\}.$$

Despite being of measure zero, badly approximable matrices form a big set in the following sense:

Theorem (Schmidt +)

The set $\mathbf{BA}_{n,m}$ has Lebesgue measure zero. This set is winning in certain games ((α, β)-game, hyperplane absolute game...). In particular, this set is dense and has full Hausdorff dimension.

Transference principle

From now on, positive decreasing to zero functions f and g will always be related by

$$g(T) = \frac{1}{f^{-1}\left(\frac{1}{T}\right)}, \qquad f(T) = \frac{1}{g^{-1}\left(\frac{1}{T}\right)}.$$

Remark

The faster f decreases, the slower g decreases.

It is due to Jarnik that:

- The asymptotic approximations of the matrix $\Theta^\top$ are related to uniform approximations of pairs of the form $(\Theta, \boldsymbol{\eta})$.
- The uniform approximations of the matrix $\Theta^\top$ are related to asymptotic approximations of pairs of the form $(\Theta, \boldsymbol{\eta})$.

New results: inhomogeneous uniform approximation

Theorem (Jarnik+)

Suppose the transposed matrix $\Theta^\top$ is not f-approximable, that is, $\Theta^\top \notin \mathbf{W}_{m,n}[f]$. Then, any pair $(\Theta, \boldsymbol{\eta})$ is $g \circ \frac{1}{\kappa^m}$-Dirichlet, where $\kappa = \kappa(n, m)$; that is,

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}\left[g \circ \frac{1}{\kappa^m}\right] = \mathbb{R}^n.$$

Theorem (Moshchevitin, N., 2025)

Suppose $\Theta^\top$ is an f-approximable matrix. If ε is small enough, then the set

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}\left[\varepsilon \cdot g \circ \frac{1}{\varepsilon}\right]$$

has Lebesgue measure zero. Works also for intersections with almost every subspace.

New results: inhomogeneous asymptotic approximation

Theorem (Jarnik+)

Suppose $\Theta^\top$ is a non f-Dirichlet matrix. Then, for any $\boldsymbol{\eta} \in \mathbb{R}^n$ the pair $(\Theta, \boldsymbol{\eta})$ is $g \circ \frac{1}{\kappa^m}$-approximable:

$$\widehat{\mathbf{W}}^{\Theta}_{n,m}\left[g \circ \frac{1}{\kappa^m}\right] = \mathbb{R}^n.$$

The next theorem provides an (nonoptimal) analog to the convergence part of Khintchine-Groshev theorem.

Theorem (Moshchevitin, N., 2025)

Suppose that Θ^T is an f-Dirichlet matrix. Let $\tilde{g}$ be an infinitely small function compared to g, defined in such a way that a certain series converges. Then the set $\widehat{\mathbf{W}}^{\Theta}_{n,m}[\tilde{g}]$ has Lebesgue measure zero. Works also for intersections with almost every subspace.

Badly approximable pairs

We define the set

$$\widehat{\mathbf{BA}}_{n,m} = \{(\Theta, \boldsymbol{\eta}) : \ ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^{n} > \frac{c}{|\mathbf{q}|^m} \ \text{ for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\}.$$

Badly approximable pairs

We define the set

$$\widehat{\mathbf{BA}}_{n,m} = \{(\Theta, \boldsymbol{\eta}) : \ ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n > \frac{c}{|\mathbf{q}|^m} \ \text{for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\}.$$

This set was widely studied before:

Badly approximable pairs

We define the set

$$\widehat{\mathbf{BA}}_{n,m} = \{(\Theta, \boldsymbol{\eta}) : \ ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n > \frac{c}{|\mathbf{q}|^m} \ \text{ for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\}.$$

This set was widely studied before:

- Kleinbock (1999): the set $\widehat{\mathbf{BA}}_{n,m}$ has full Hausdorff dimension;

Badly approximable pairs

We define the set

$$\widehat{\mathbf{BA}}_{n,m} = \{(\Theta, \boldsymbol{\eta}) : \ ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n > \frac{c}{|\mathbf{q}|^m} \text{ for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\}.$$

This set was widely studied before:

- Kleinbock (1999): the set $\widehat{\mathbf{BA}}_{n,m}$ has full Hausdorff dimension;
- Einsiedler, Tseng (2011): the sets $\widehat{\mathbf{BA}}_{n,m}^{\Theta}$ and $\widehat{\mathbf{BA}}_{n,m}^{\boldsymbol{\eta}}$ are winning on certain fractals (also similar independent results by Broderick, Fishman and Kleinbock);

Badly approximable pairs

We define the set

$$\widehat{\mathbf{BA}}_{n,m} = \{(\Theta, \boldsymbol{\eta}) : \ ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n > \frac{c}{|\mathbf{q}|^m} \ \text{ for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\}.$$

This set was widely studied before:

- Kleinbock (1999): the set $\widehat{\mathbf{BA}}_{n,m}$ has full Hausdorff dimension;
- Einsiedler, Tseng (2011): the sets $\widehat{\mathbf{BA}}_{n,m}^{\Theta}$ and $\widehat{\mathbf{BA}}_{n,m}^{\boldsymbol{\eta}}$ are winning on certain fractals (also similar independent results by Broderick, Fishman and Kleinbock);
- Broderick, Fishman, Simmons (2013): the sets $\widehat{\mathbf{BA}}_{n,m}^{\Theta}$ are HAW (in particular, winning on certain fractals).

Badly approximable pairs

We define the set

$$\widehat{\mathbf{BA}}_{n,m} = \{(\Theta, \boldsymbol{\eta}) : \ ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n > \frac{c}{|\mathbf{q}|^m} \ \text{ for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\}.$$

This set was widely studied before:

- Kleinbock (1999): the set $\widehat{\mathbf{BA}}_{n,m}$ has full Hausdorff dimension;
- Einsiedler, Tseng (2011): the sets $\widehat{\mathbf{BA}}_{n,m}^{\Theta}$ and $\widehat{\mathbf{BA}}_{n,m}^{\boldsymbol{\eta}}$ are winning on certain fractals (also similar independent results by Broderick, Fishman and Kleinbock);
- Broderick, Fishman, Simmons (2013): the sets $\widehat{\mathbf{BA}}_{n,m}^{\Theta}$ are HAW (in particular, winning on certain fractals).

It is also known:

- Kim (2024): if Θ is nonsingular, then $\widehat{\mathbf{BA}}_{n,m}^{\Theta}$ is of measure zero.

g-Badly approximable pairs

We now introduce the natural generalization:

$$\widehat{\mathbf{BA}}_{n,m}[g] = \{(\Theta, \boldsymbol{\eta}) : \ ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n > cg(|\mathbf{q}|^m) \ \text{ for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\};$$

Easy to see: $\widehat{\mathbf{BA}}_{n,m} = \widehat{\mathbf{BA}}_{n,m}[f_1]$.

g-Badly approximable pairs

We now introduce the natural generalization:

$$\widehat{\mathbf{BA}}_{n,m}[g] = \{(\Theta, \boldsymbol{\eta}) : \ \ ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n > cg(|\mathbf{q}|^m) \ \text{ for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\};$$

Easy to see: $\widehat{\mathbf{BA}}_{n,m} = \widehat{\mathbf{BA}}_{n,m}[f_1]$.

- Moshchevitin (2011): if f satisfies the technical condition and $\Theta^\top \in \mathbf{D}_{m,n}[f]$ is an f-Dirichlet matrix, then the set

$$\widehat{\mathbf{BA}}^{\Theta}_{n,m}[g]$$

is winning.

Technical condition

There exists $\gamma > 0$ that for any $T \in \mathbb{R}_+$ and any $C > 1$ large enough, one has

$$f(CT) \leq C^{-\gamma} f(T).$$

g-Badly approximable pairs

Theorem (Moshchevitin, N., 2025)

Suppose $\Theta^\top$ is a non f-Dirichlet matrix. Then, the set

$$\widehat{\mathbf{BA}}^{\Theta}_{n,m}\left[g \circ \frac{1}{\kappa^m}\right]$$

of $g \circ \frac{1}{\kappa^m}$-Badly approximable shift vectors $\boldsymbol{\eta}$ has measure zero.

Theorem (Moshchevitin, N., 2025)

Suppose f satisfies the technical condition and $\Theta^\top \in \mathbf{D}_{m,n}[f]$ is an f-Dirichlet matrix. Then, the set $\widehat{\mathbf{BA}}^{\Theta}_{n,m}[g]$ is HAW. Works also for intersections with certain subspaces.

Remark

Normally $\widehat{\mathbf{BA}}^{\Theta}_{n,m}[g] \subseteq \widehat{\mathbf{BA}}^{\Theta}_{n,m} = \widehat{\mathbf{BA}}^{\Theta}_{n,m}[f_1]$

Applications and examples

Bugeaud, Laurent (2005) + Moshchevitin, N. (2025)

For any $\boldsymbol{\eta} \in \mathbb{R}^n$,

$$\omega(\Theta, \boldsymbol{\eta}) \geq \frac{1}{\hat{\omega}(\Theta^\top)} \qquad \text{and} \qquad \hat{\omega}(\Theta, \boldsymbol{\eta}) \geq \frac{1}{\omega(\Theta^\top)}. \tag{2}$$

Moreover, for almost every affine subspace $\mathcal{A} \subseteq \mathbb{R}^n$:
For Lebesgue-almost all $\boldsymbol{\eta} \in \mathcal{A}$, the equality in (2) holds.

Similar statements hold if we take functions of the form $T^a \log T^b \log\log T^c \ldots$ instead of just powers of T.

Applications and examples

Definition

The matrix Θ is called very well approximable ($\mathbf{VWA}_{m,n}$) if there exists $\varepsilon > 0$, such that

$$\Theta \in \mathbf{W}_{m,n}[f_{1+\varepsilon}].$$

- If Θ is not very well approximable, then for any $\varepsilon > 0$ and any $\boldsymbol{\eta}$ the pair $(\Theta, \boldsymbol{\eta})$ is $f_{1-\varepsilon}$-Dirichlet:

$$\widehat{\mathbf{D}}^{\Theta}_{m,n}[f_{1-\varepsilon}] = \mathbb{R}^n.$$

- If Θ is very well approximable, then for almost every $\boldsymbol{\eta}$ there exists $\varepsilon > 0$:

$$(\Theta, \boldsymbol{\eta}) \notin \widehat{\mathbf{D}}_{m,n}[f_{1-\varepsilon}].$$

In addition, this set of $\boldsymbol{\eta}$ of full measure is HAW.

Dirichlet's theorem for pairs

Inhomogeneous Dirichlet's theorem

Theorem (Kleinbock and Wadleigh, 2019)

Let $g(z),\ \ z \geq 1$ be a non-increasing real valued function. The set $\widehat{\mathbf{D}}_{n,m}\,[g]$ has zero measure if the following series diverges, and full measure if it converges:

$$\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)}$$

Inhomogeneous Dirichlet's theorem

Theorem (Kleinbock and Wadleigh, 2019)

Let $g(z),\ \ z \geq 1$ be a non-increasing real valued function. The set $\widehat{\mathbf{D}}_{n,m}\left[g\right]$ has zero measure if the following series diverges, and full measure if it converges:

$$\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)}$$

Theorem (Khintchine–Groshev)

Let $f(t),\ \ t \geq 1$ be a non-increasing real valued function. The set $\mathbf{W}_{n,m}\left[f\right]$ has zero measure if the following series converges, and full measure if it diverges:

$$\sum_{k=1}^{\infty} f(k)$$

Proof of equivalence

- As usual,

$$g(T) = \frac{1}{f^{-1}\left(\frac{1}{T}\right)}, \qquad f(T) = \frac{1}{g^{-1}\left(\frac{1}{T}\right)}.$$

- $$\int_{1}^{+\infty} \frac{dz}{z^2 g(z)} \qquad \text{and} \qquad \int_{1}^{+\infty} f(t)dt$$

  either both converge or both diverge.

Proof of equivalence: convergent case

- Suppose $\sum\limits_{j=1}^{\infty} \frac{1}{j^2 g(j)} < \infty$. By Khintchine-Groshev's theorem then for almost every Θ,

$$\Theta^\top \notin \mathbf{W}_{n,m}[f];$$

and thus

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}\left[g \circ \frac{1}{\kappa^m}\right] = \mathbb{R}^n.$$

So almost every pair is $g \circ \frac{1}{\kappa^m}$-Dirichlet.

Proof of equivalence: divergent case

- Suppose $\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)} = \infty$. By Khintchine-Groshev's theorem then for almost every Θ,

$$\Theta^\top \in \mathbf{W}_{n,m}[f];$$

and so

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}\left[\varepsilon \cdot g \circ \frac{1}{\varepsilon}\right]$$

has zero measure. Therefore almost no pairs are $\varepsilon \cdot g \circ \frac{1}{\varepsilon}$-Dirichlet.

Weighted case

$$\sum_{i=1}^{n} \rho_i = \sum_{j=1}^{m} \sigma_j = 1.$$

Define the quasinorms $|\cdot|_\rho$ and $|\cdot|_\sigma$ on $\mathbb{R}^n$ and $\mathbb{R}^m$ by

$$|\mathbf{y}|_\rho = \max_{i=1,\ldots,n} |y_i|^{\frac{1}{\rho_i}} \qquad \text{and} \qquad |\mathbf{x}|_\sigma = \max_{j=1,\ldots,m} |x_j|^{\frac{1}{\sigma_j}}.$$

Let $\widehat{\mathbf{D}}_{n,m}[g;\rho,\sigma]$ be the set of g-Dirichlet pairs with respect to weighted quasinorms.

Theorem (To appear)

Let $g(z),\ \ z \geq 1$ be a non-increasing real valued function. The set $\widehat{\mathbf{D}}_{n,m}[g;\rho,\sigma]$ has zero measure if the following series diverges, and full measure if it converges:

$$\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)}$$

Best approximations

Irrationality measure functions

We consider the ordinary irrationality measure function

$$\psi_\Theta(t) = \min_{\mathbf{q}=(q_1,\ldots,q_m)\in\mathbb{Z}^m:\, 0<|\mathbf{q}|\leq t} ||\Theta\mathbf{q}||_{\mathbb{Z}^n},$$

and the irrationality measure function for inhomogeneous approximations

$$\psi_{\Theta,\boldsymbol{\eta}}(t) = \min_{\mathbf{q}\in\mathbb{Z}^m:\, 0<|\mathbf{q}|\leq t} ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}.$$

Best approximation vectors

Definition

We call Θ trivially singular if its columns together with integer vectors are linearly dependent over $\mathbb{Q}$.

If Θ is not trivially singular, there exists the sequence

$$\mathbf{y}_\nu \in \mathbb{Z}^m, \;\; Y_\nu = |\mathbf{y}_\nu|, \;\; \mathbf{a}_\nu \in \mathbb{Z}^n, \;\;\; Y_1 = 1 < Y_2 < \ldots < Y_\nu < Y_{\nu+1} < \ldots,$$

$$\psi_\Theta(t) = \psi_\Theta(Y_\nu) = ||\Theta \mathbf{y}_\nu||_{\mathbb{Z}^n} = |\Theta \mathbf{y}_\nu - \mathbf{a}_\nu|, \quad \text{for} \;\; Y_\nu \le t < Y_{\nu+1}$$

of the best approximations.

A useful lemma

Here $A = A(n, m)$ and $B = B(n, m)$ are some known constants.

Lemma

For any Θ which is not trivially singular and for any ν one has

1. $Y_{\nu+A} \geq 2Y_\nu$ (Bugeaud, Laurent) and
2. $\psi_\Theta(Y_{\nu+B}) \leq \frac{1}{2}\psi_\Theta(Y_\nu)$ (Moshchevitin, N.)

Some transference lemmata

Basic observation

Let $\{\mathbf{y}_\nu\}$ be the sequence of best approximations for $\Theta^\top$ and $Y_\nu = |\mathbf{y}_\nu|$. Note that

$$\boldsymbol{\eta} \cdot \mathbf{y}_\nu = \sum_{j=1}^{m} q_j \Theta_j^T(\mathbf{y}_\nu) - \sum_{i=1}^{n} (\Theta_i(\mathbf{q}) - \eta_i)\, y_{\nu,i},$$

and so

$$||\boldsymbol{\eta} \cdot \mathbf{y}_\nu|| \leq m|\mathbf{q}|\psi_{\Theta^\top}(Y_\nu) + nY_\nu \cdot ||\Theta\mathbf{q} - \boldsymbol{\eta}||.$$

Main transference lemma

Lemma (Cassels)

Let M and X be two positive real numbers, such that the inequality

$$||\Theta^\top \mathbf{y}|| \geq \frac{\kappa}{X}$$

holds for any nonzero $\mathbf{y} \in \mathbb{Z}^n$ with $|\mathbf{y}| \leq M$. Then for any $\boldsymbol{\eta} \in \mathbb{R}^n$ there exists such a $\mathbf{q} \in \mathbb{Z}^m$ such that

$$\begin{cases} |\mathbf{q}| \leq X \\ ||\Theta \mathbf{q} - \boldsymbol{\eta}|| \leq \frac{\kappa}{M}. \end{cases}$$

Main transference lemma

Lemma (Cassels)

Let M and X be two positive real numbers, such that the inequality

$$||\Theta^\top \mathbf{y}|| \geq \frac{\kappa}{X}$$

holds for any nonzero $\mathbf{y} \in \mathbb{Z}^n$ with $|\mathbf{y}| \leq M$. Then for any $\boldsymbol{\eta} \in \mathbb{R}^n$ there exists such a $\mathbf{q} \in \mathbb{Z}^m$ such that

$$\begin{cases} |\mathbf{q}| \leq X \\ ||\Theta\mathbf{q} - \boldsymbol{\eta}|| \leq \frac{\kappa}{M}. \end{cases}$$

We can strengthen the second inequality if we only want it to hold for *almost all* $\boldsymbol{\eta}$:

Metric version of transference lemma

Lemma (Moshchevitin, N., 2025)

Let $\{\psi_k\}$ be a sequence of positive real numbers, $\lim\limits_{k\to\infty} \psi_k = 0$ and $\sum\limits_{k=1}^{\infty} \psi_k^n = \infty$.

Let $\{M_k\}$, $\{X_k\}$ be two sequences of real numbers such that the inequality

$$||\Theta^\top \mathbf{y}|| \geq \frac{\kappa}{X_k}$$

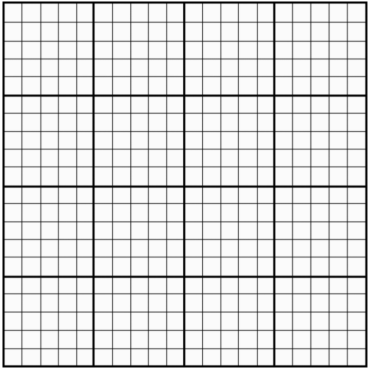
holds for any nonzero $\mathbf{y} \in \mathbb{Z}^n$ with $|\mathbf{y}| < M_k$, and in addition $\lim\limits_{k\to\infty} \psi_k \frac{M_{k+1}}{M_k} = \infty$.

Then, for almost every $\boldsymbol{\eta} \in \mathbb{R}^n$ there exists infinitely many $k \in \mathbb{N}$, such that the system of inequalities has a solution $\mathbf{q} \in \mathbb{Z}^m$:

$$\begin{cases} |\mathbf{q}| \leq X_k \\ ||\Theta \mathbf{q} - \boldsymbol{\eta}|| \leq \frac{\kappa \psi_k}{M_k} \end{cases}$$

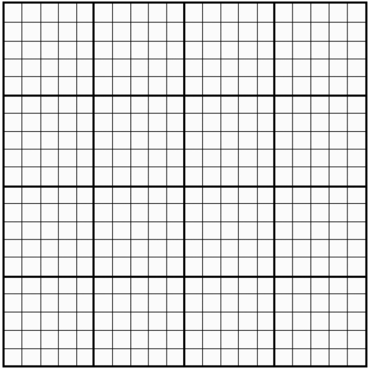
Proof overview

- Fix k and cover the cube $[0,1)^n$ by $W_k \sim \left(\frac{M_k}{\kappa}\right)^n$ boxes with side length $\frac{2\kappa}{M_k}$.



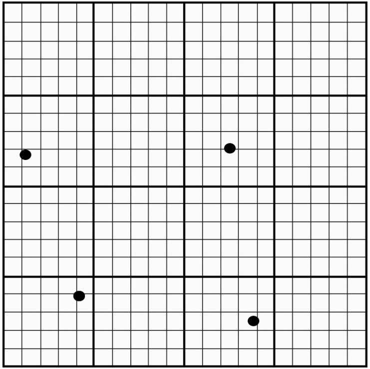
Proof overview

- Fix k and cover the cube $[0,1)^n$ by $W_k \sim \left(\frac{M_k}{\kappa}\right)^n$ boxes with side length $\frac{2\kappa}{M_k}$.
- By Cassels' lemma, each of these boxes contains a point of the form $\Theta\mathbf{q}$ where $|\mathbf{q}| \leq X_k$.



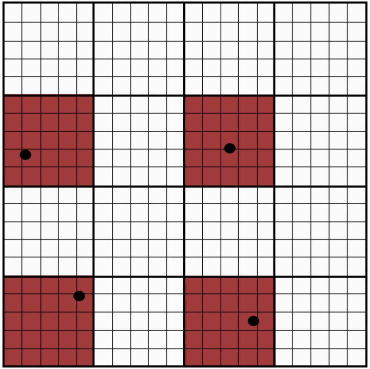
Proof overview

- Fix k and cover the cube $[0,1)^n$ by $W_k \sim \left(\frac{M_k}{\kappa}\right)^n$ boxes with side length $\frac{2\kappa}{M_k}$.
- By Cassels' lemma, each of these boxes contains a point of the form $\Theta\mathbf{q}$ where $|\mathbf{q}| \leq X_k$.



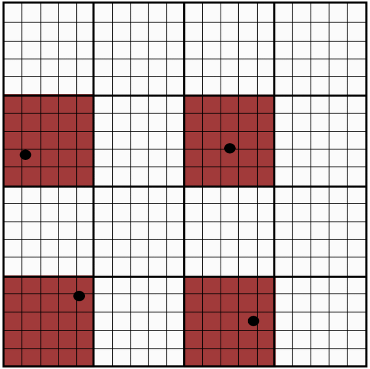
Proof overview

- Consider the boxes with all the odd indexes; there are $W_k' \sim W_k \sim \left(\frac{M_k}{\kappa}\right)^n$ of them.



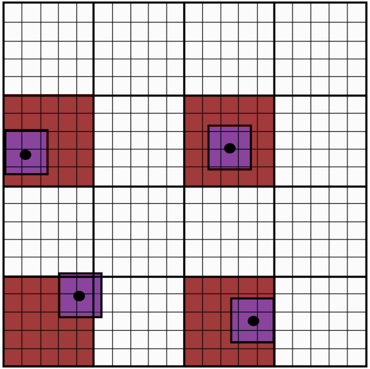
Proof overview

- Consider the boxes with all the odd indexes; there are $W'_k \sim W_k \sim \left(\frac{M_k}{\kappa}\right)^n$ of them.
- Pick a point of the form $\Theta\mathbf{q}$ in each of them and center a box with a side $\frac{2\psi_k\kappa}{M_k}$ at this point. Note that these smaller boxes are disjoint.



Proof overview

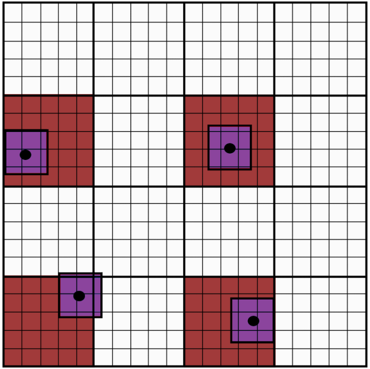
- Consider the boxes with all the odd indexes; there are $W_k' \sim W_k \sim \left(\frac{M_k}{\kappa}\right)^n$ of them.
- Pick a point of the form $\Theta\mathbf{q}$ in each of them and center a box with a side $\frac{2\psi_k\kappa}{M_k}$ at this point. Note that these smaller boxes are disjoint.



Proof overview

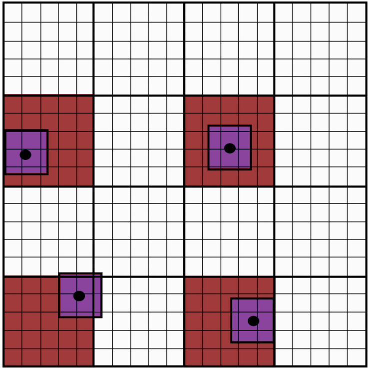
- Let E_k be the union of all the small boxes; one has

$$m(E_k) \gg \psi_k^n;$$



Proof overview

- Let E_k be the union of all the small boxes; one has
$$m(E_k) \gg \psi_k^n;$$
- As $\frac{1}{M_k} \cdot \frac{M_k}{\psi_k} \to 0$, the sets E_k become "almost independent" as k grows:
$$m(E_k \cap E_s) \leq (1+\varepsilon) m(E_s) m(E_k)$$
for s and k large enough.



Proof overview

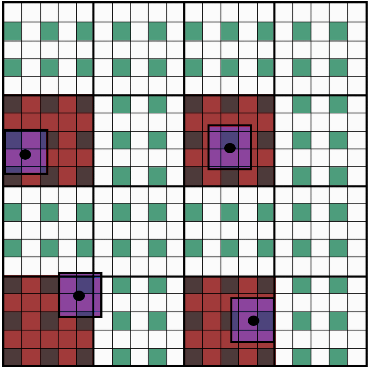
- Let E_k be the union of all the small boxes; one has

$$m(E_k) \gg \psi_k^n;$$

- As $\frac{1}{M_{k+1}} \cdot \frac{M_k}{\psi_k} \to 0$, the sets E_k become "almost independent" as k grows:

$$m(E_k \cap E_s) \leq (1+\varepsilon) m(E_s) m(E_k)$$

for s and k large enough.



Proof overview

It remains to recall a classical statement:

Lemma

Let X be a measure space with measure μ, and let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets. Let $E = \bigcap\limits_{N=1}^{\infty} \bigcup\limits_{k=N}^{\infty} E_k$. Suppose $\sum\limits_{k=1}^{\infty} \mu(E_k) = \infty$. Then,

$$\mu(E) \geq \limsup_{N\to\infty} \frac{\left(\sum\limits_{k=1}^{N} \mu(E_k)\right)^2}{\sum\limits_{k,s=1}^{N} \mu(E_k \cap E_s)}.$$

It remains to notice that in our case we guarantee $RHS \geq 1 - \varepsilon$ for any $\varepsilon > 0$; and that $\limsup\limits_{k\to\infty} E_k$ is the desired set.

Exceptional subspaces

Ξ-Exceptional subspaces

Let $\{\mathbf{y}_\nu\}$ be the sequence of best approximations for $\Theta^\top$ and $Y_\nu = |\mathbf{y}_\nu|$.

Definition

We call a point $\mathbf{v} \in \mathbb{R}^n$ on the unit sphere (direction) Ξ-*exceptional* (for the matrix $\Theta^\top$) if for any $\delta > 0$ there exist infinitely many $\nu \in \mathbb{N}$ for which $\widehat{\mathbf{y}_\nu, \mathbf{v}} < \frac{\delta}{|\mathbf{y}_\nu|^\Xi}$.

Definition

Let $\mathcal{A}$ be an affine subspace of $\mathbb{R}^n$ and $\mathcal{L}$ — the corresponding linear subspace. We call $\mathcal{A}$ Ξ-*exceptional* if $\mathcal{L}^\perp$ contains Ξ-exceptional points.

Ξ-Exceptional subspaces

Informally, non-exceptional subspaces are those where the projections of $\mathbf{y}_\nu$ grow fast enough.

Lemma

Let $\Xi > 0$ and X_E be the set of k-dimensional Ξ-exceptional linear subspaces of $\mathbb{R}^n$.
Then, $\dim_H X_E = k(n-k-1)$, if not empty.
In particular, almost no affine subspaces are Ξ-exceptional.

Most of our metrical results work also on any non-exceptional subspaces.

Asymptotic directions

Let $\Theta^\top$ be a non trivially singular matrix, and $\mathbf{y}_\nu$ its sequence of the best approximation vectors. Let S^{n-1} be the $n-1$-dimensional sphere in $\mathbb{R}^n$ centered at zero.

Definition

We say ζ is an *asymptotic direction* for the best approximations sequence $\mathbf{y}_\nu$ if ζ is a limit point of the set $\left\{\pm\frac{\mathbf{y}_\nu}{|\mathbf{y}_\nu|} : \nu \in \mathbb{N}\right\}$.

Lemma (German, 2004)

Let $\Omega \subseteq S^{n-1}$ be a closed centrally symmetric set. Then, the set of $1 \times n$ matrices (linear forms) $\Theta^\top$ for which Ω is the set of asymptotic directions has cardinality continuum.

Our results covering the winning properties (HAW) also work on any affine subspaces, not orthogonal to any asymptotic directions of $\Theta^\top$.

Work in progress

- Weighted setup, and other quasinorms (to appear soon);
- Hausdorff dimensions of the sets $\widehat{\mathbf{D}}^{\Theta}_{m,n}[g]$ and $\widehat{\mathbf{W}}^{\Theta}_{m,n}[g]$;
- Optimal condition for nullity of $\widehat{\mathbf{W}}^{\Theta}_{m,n}[g]$.

Thank you for your attention!