Zero-one laws for uniform inhomogeneous Diophantine approximations

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Introduction

Notation

We will use the following conventions:

- m and n are positive integers and $d = m + n$. $\mathbf{M}_{n,m}$ is the set of all real $n \times m$ matrices.

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  — a pair of a real $n \times m$ matrix and n-vector.

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  — a pair of a real $n \times m$ matrix and n-vector.
- $|\cdot|$ is the supremum norm of a vector, and $||\cdot||_{\mathbb{Z}^n}$ is the distance to the nearest integer vector function.

Notation

- Classical homogeneous Diophantine approximations studies the form

$$||\Theta\mathbf{q}||_{\mathbb{Z}^n}$$

- In the inhomogeneous setting, we replace the linear for with affine: we study

$$||\Theta\mathbf{q}-\boldsymbol{\eta}||_{\mathbb{Z}^n}$$

More specifically, we want to compare the values of this forms with values of certain functions:

Uniform and asymptotic approximations

Fix a non-increasing real function g. We study the system of inequalities

$$\begin{cases} ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n \leq g(T) \\ |\mathbf{q}|^m \leq T \end{cases} \tag{1}$$

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In both homogeneous and inhomogeneous setups, two types of questions are of interest:

- Uniform approximations: does the system (1) have an integer solution $\mathbf{q}$ for all T large enough? $\widehat{\mathbf{D}}_{n,m}[g]$
- Asymptotic approximations: does the system (1) have an integer solution $\mathbf{q}$ for an unbounded set of T? $\widehat{\mathbf{W}}_{n,m}[g]$

Main definitions

Definition (g-Dirichlet pairs)

By $\widehat{\mathbf{D}}_{n,m}[g]$ we will denote the set of pairs $(\Theta, \boldsymbol{\eta})$ for which the system

$$\begin{cases} ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n \leq g(T) \\ |\mathbf{q}|^m \leq T \end{cases}$$

has an integer solution $\mathbf{q}$ for all T large enough. *Homogeneous case:* $\mathbf{D}_{n,m}[g]$.

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Definition (g-approximable pairs)

By $\widehat{\mathbf{W}}_{n,m}[g]$ we will denote the set of pairs $(\Theta, \boldsymbol{\eta})$ for which the inequality

$$||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\mathbb{Z}^n}^n \leq g(|\mathbf{q}|^m)$$

has infinitely many integer solutions $\mathbf{q}$. *Homogeneous case:* $\mathbf{W}_{n,m}[g]$.

General weight functions

Definition

Let

$$\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \quad \text{and} \quad \underline{\beta} = (\beta_1, \ldots, \beta_m)$$

be the n-tuple and m-tuple of positive, increasing, continuous real valued functions, defined on the interval $[1, +\infty)$ and having infinite limit an infinity. It will be convenient to assume that $\alpha_i(1) = \beta_j(1) = 1$. One can extend these functions to the whole set of nonnegative real arguments by defining

$$\alpha_i(T) = \frac{1}{\alpha_i\left(\frac{1}{T}\right)} \qquad \text{and} \qquad \beta_j(T) = \frac{1}{\beta_j\left(\frac{1}{T}\right)}$$

for $0 < T \leq 1$, and, by continuity, $\alpha_i(0) = \beta_j(0) = 0$.

General weight functions

For $\mathbf{y} = (y_1, \ldots, y_n)$ and $\mathbf{x} = (x_1, \ldots, x_m)$, we write

$$|\mathbf{y}|_{\underline{\alpha}} = \max_{1 \le i \le n} \alpha_i^{-1}(|y_i|) \qquad \text{and} \qquad |\mathbf{x}|_{\underline{\beta}} = \max_{1 \le j \le m} \beta_j^{-1}(|x_j|).$$

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By $||\cdot||_{\underline{\alpha}}$ and $||\cdot||_{\underline{\beta}}$ we will denote the corresponding "distance to the nearest integer" functions, that is,

$$||\mathbf{y}||_{\underline{\alpha}} = \min_{\mathbf{p} \in \mathbb{Z}^n} |\mathbf{y} - \mathbf{p}|_{\underline{\alpha}} \qquad \text{and} \qquad ||\mathbf{x}||_{\underline{\beta}} = \min_{\mathbf{q} \in \mathbb{Z}^n} |\mathbf{x} - \mathbf{q}|_{\underline{\beta}}.$$

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Example

Let $\alpha_i(T) = \beta_j(T) = T$. Then $|\mathbf{y}|_{\underline{\alpha}} = |\mathbf{y}|$ and $|\mathbf{x}|_{\underline{\beta}} = |\mathbf{x}|$.

Approximations with general weight functions

Definition

1. $\Theta^\top$ is $(f, \underline{\beta}, \underline{\alpha})$-approximable if the inequality

$$||\Theta^\top \mathbf{y}||_{\underline{\beta}} \leq f(|\mathbf{y}|_{\underline{\alpha}})$$

has infinitely many solutions $\mathbf{y} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. $\mathbf{W}_{m,n}[f; \underline{\beta}, \underline{\alpha}]$

2. $(\Theta, \boldsymbol{\eta})$ is $(g, \underline{\alpha}, \underline{\beta})$-Dirichlet if

$$\begin{cases} ||\Theta\mathbf{q} - \boldsymbol{\eta}||_{\underline{\alpha}} \leq g(T), \\ |\mathbf{q}|_{\underline{\beta}} \leq T \end{cases}$$

has a solution $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ for any $T \in \mathbb{R}_+$ large enough. $\widehat{\mathbf{D}}_{n,m}[g; \underline{\alpha}, \underline{\beta}]$

Approximations with general weight functions

1. $\mathbf{W}_{m,n}[f;\underline{\beta},\underline{\alpha}]$:

$$\begin{cases} \|\left(\Theta^\top\right)_j \mathbf{y}\| \le \beta_j(f(T)), & j = 1,\dots,m; \\ |y_i| \le \alpha_i(T), & i = 1,\dots,n \end{cases}$$

has a solution $\mathbf{y} = (y_1,\dots,y_n) \in \mathbb{Z}^n$ for an unbounded set of $T \in \mathbb{R}_+$.

2. $\widehat{\mathbf{D}}_{n,m}[g;\underline{\alpha},\underline{\beta}]$:

$$\begin{cases} \|\Theta_i\mathbf{q} - \eta_i\| \le \alpha_i(g(T)), & i = 1,\dots,n; \\ |q_j| \le \beta_j(T), & j = 1,\dots,m \end{cases}$$

has a solution $\mathbf{q} = (q_1,\dots,q_m) \in \mathbb{Z}^m$ for any $T \in \mathbb{R}_+$ large enough.

Quasimultiplicativity

An increasing function $h : \ \mathbb{R}_+ \to \mathbb{R}_+$ is *quasimultiplicative* if $\exists M > 1 : \ \forall |k| \gg 1$

$$c_1 h(M^k) \leq h(M^{k+1}) \leq c_2 h(M^k), \tag{2}$$

where $1 < c_1 \leq c_2$ are some absolute constants.

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- $h + r$, $h \cdot r$, $h \circ r$ and $r \circ h$ are quasimultiplicative functions.
- $\frac{1}{h}$ and h^{-1} are quasimultiplicative.

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- $h + r$, $h \cdot r$, $h \circ r$ and $r \circ h$ are quasimultiplicative functions.
- $\frac{1}{h}$ and h^{-1} are quasimultiplicative.
- if h is a quasimultiplicative function for which

$$h(T) = \frac{1}{h\left(\frac{1}{T}\right)} \quad \text{for all } \ T \in \mathbb{R}_+,$$

  then it is enough to check condition (2) only for positive values of k.

Examples: approximations with weights

Let

$$\sum_{i=1}^{n} \rho_i = \sum_{j=1}^{m} \sigma_j = 1;$$

define

$$\alpha_i(T) = T^{\rho_i}, \qquad \beta_j(T) = T^{\sigma_j}.$$

These functions define the quasinorms $|\cdot|_\rho$ and $|\cdot|_\sigma$ on $\mathbb{R}^n$ and $\mathbb{R}^m$ by

$$|\mathbf{y}|_\rho = \max_{i=1,\dots,n} |y_i|^{\frac{1}{\rho_i}} \qquad \text{and} \qquad |\mathbf{x}|_\sigma = \max_{j=1,\dots,m} |x_j|^{\frac{1}{\sigma_j}}.$$

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We use the notation: $\widehat{\mathbf{D}}[g;\rho,\sigma]$, $\widehat{\mathbf{W}}[g;\rho,\sigma]$.

Nontrivial example

Let $\theta_1, \theta_2, \eta \in \mathbb{R}$. Consider the system

$$\begin{cases} ||\theta_1 q_1 + \theta_2 q_2 - \eta|| \leq g(T), \\ |q_1| \leq \beta_1(T), \\ |q_2| \leq \beta_2(T), \end{cases}$$

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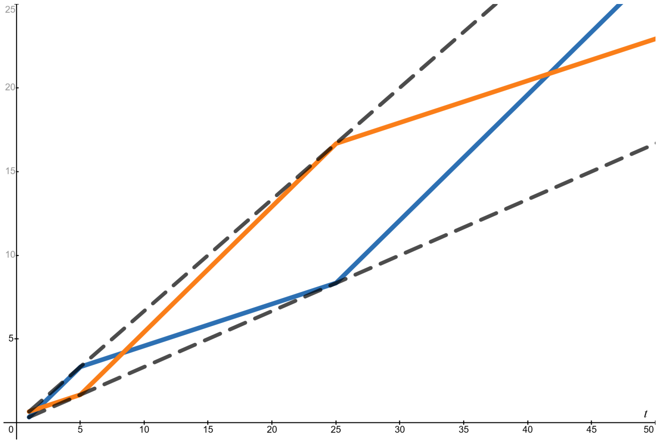
We construct a pair of quasimultiplicative functions β_1, β_2 such that:

- For an unbounded set of values of T, $\beta_1(T) = T^{\frac{1}{3}}$ and $\beta_2(T) = T^{\frac{2}{3}}$;
- For an unbounded set of values of T, $\beta_1(T) = T^{\frac{2}{3}}$ and $\beta_2(T) = T^{\frac{1}{3}}$.

Nontrivial example



Orange and blue functions are the functions β_j on logarithmic scale (slopes equal $\frac{3}{4}$ or $\frac{1}{4}$).

Sections of sets of pairs

In the inhomogeneous case we can study three types of objects:

- Sets of pairs $(\Theta, \boldsymbol{\eta})$, satisfying some restrictions: $\widehat{\mathbf{D}}_{n,m}\left[g; \underline{\alpha}, \underline{\beta}\right]$, $\widehat{\mathbf{W}}_{n,m}\left[g; \underline{\alpha}, \underline{\beta}\right]$.
- Sets of matrices Θ for a fixed "shift vector" $\boldsymbol{\eta}$ ("sections with a fixed vector"): $\widehat{\mathbf{D}}^{\boldsymbol{\eta}}_{n,m}\left[g; \underline{\alpha}, \underline{\beta}\right]$, $\widehat{\mathbf{W}}^{\boldsymbol{\eta}}_{n,m}\left[g; \underline{\alpha}, \underline{\beta}\right]$.
- Sets of vectors $\boldsymbol{\eta}$ for a fixed matrix Θ ("sections with a fixed matrix"): $\widehat{\mathbf{D}}^{\Theta}_{n,m}\left[g; \underline{\alpha}, \underline{\beta}\right]$, $\widehat{\mathbf{W}}^{\Theta}_{n,m}\left[g; \underline{\alpha}, \underline{\beta}\right]$.

Throughout the talk functions α_i, β_j are quasimultiplicative. Function g is always a non increasing real valued function.

Overview of results

Homogeneous uniform approximation

We will use the notation: $f_a(T) = \frac{1}{T^a}$, in particular, $f_1(T) = \frac{1}{T}$.

Theorem (Dirichlet)

Any matrix $\Theta \in \mathbf{M}_{n,m}$ is f_1-Dirichlet:

$$\mathbf{D}_{n,m}[f_1] = \mathbf{M}_{n,m}.$$

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$$\mathbf{D}_{n,m}\left[f_1\right] = \mathbf{M}_{n,m}.$$

Let $\alpha(T) := \prod_{i=1}^{n} \alpha_i(T)$ and $\beta(T) := \prod_{j=1}^{m} \beta_j(T)$, and define the function $g_{\underline{\alpha},\underline{\beta}}(T) := \alpha^{-1}\left(\frac{1}{\beta(T)}\right)$.

Theorem (Simple observation)

Any matrix $\Theta \in \mathbf{M}_{n,m}$ is $(g_{\underline{\alpha},\underline{\beta}}, \underline{\alpha}, \underline{\beta})$-Dirichlet, and thus is $(g_{\underline{\alpha},\underline{\beta}}, \underline{\alpha}, \underline{\beta})$-approximable.

Dirichlet improvable matrices have measure zero

Dirichlet's theorem provides an "for all" bound. It turns out, however, that this bound is also metrically optimal. More precisely:

Theorem (Davenport, Schmidt, 1970)

If $\varepsilon < 1$, the set of ε-Dirichlet improvable matrices has Lebesgue measure zero:

$$m\left(\mathbf{D}_{n,m}\left[\varepsilon f_1\right]\right) = 0.$$

Transference principle

From now on, positive decreasing to zero functions f and g will always be related by

$$g(T) = \frac{1}{f^{-1}\left(\frac{1}{T}\right)}, \qquad f(T) = \frac{1}{g^{-1}\left(\frac{1}{T}\right)}.$$

Remark

The faster f decreases, the slower g decreases.

It is due to Jarnik that:

- The asymptotic approximations of the matrix $\Theta^\top$ are related to uniform approximations of pairs of the form $(\Theta, \boldsymbol{\eta})$.
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New results: inhomogeneous uniform approximation

Theorem (Jarnik+)

Suppose the transposed matrix $\Theta^\top$ *is not* f*-approximable, that is,* $\Theta^\top \notin \mathbf{W}_{m,n}[f]$.
Then, any pair $(\Theta, \boldsymbol{\eta})$ *is* $g \circ \frac{1}{\kappa^m}$*-Dirichlet, where* $\kappa = \kappa(n, m)$*; that is,*

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}\left[g \circ \frac{1}{\kappa^m}\right] = \mathbb{R}^n.$$

Theorem (Moshchevitin, N., 2025)

Suppose $\Theta^\top$ *is an* f*-approximable matrix. If* ε *is small enough, then the set*

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}\left[\varepsilon \cdot g \circ \frac{1}{\varepsilon}\right]$$

has Lebesgue measure zero. Works also for intersections with almost every subspace.

New results: inhomogeneous uniform approximation

Theorem (Jarnik+, 2025)

Suppose f is such a continuous, strictly decreasing function that $\Theta^T \notin \mathbf{W}_{m,n}[f;\underline{\beta},\underline{\alpha}]$. Let $C = C_d := d \cdot d!$, where $d = n+m$. Then,

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}\left[g; C\underline{\alpha}, C\underline{\beta}\right] = \mathbb{R}^n.$$

Theorem (N., 2025)

Suppose f is such that $\Theta^T \in \mathbf{W}_{m,n}[f;\underline{\beta},\underline{\alpha}]$. If ε is small enough, then the set

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}[\varepsilon \cdot g \circ \frac{1}{\varepsilon}; \underline{\alpha}, \underline{\beta}]$$

has Lebesgue measure zero. Works also for intersections with almost every subspace.

New results: inhomogeneous uniform approximation

For the next statement we introduce a natural generalization of singularity.

Definition

$$\widehat{\mathbf{Sing}}_{n,m}[g] = \bigcap_{\varepsilon>0} \widehat{\mathbf{D}}[\varepsilon g].$$

The "Twisted Davenport-Schmidt's theorem" shows: if $\Theta^\top$ is f-approximable, then the set $\widehat{\mathbf{Sing}}^{\Theta}_{n,m}[g]$ has measure zero. The statement below strengthens this statement.

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Theorem (Moshchevitin, N., 2025)

Suppose f satisfies an additional technical condition, and $\Theta^\top$ is an f-approximable matrix. Then the set

$$\left(\widehat{\mathbf{Sing}}^{\Theta}_{n,m}[g]\right)^c$$

is HAW; in particular, is winning in Schmidt's game.

Dirichlet's theorem for pairs

Inhomogeneous Dirichlet's theorem

Theorem (Kleinbock and Wadleigh, 2019)

The set $\widehat{\mathbf{D}}_{n,m}[g]$ *has zero measure if the following series diverges, and full measure if it converges:*

$$\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)}$$

Inhomogeneous Dirichlet's theorem

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Theorem (N., 2025)

The set $\widehat{\mathbf{D}}_{n,m}[g;\underline{\alpha},\underline{\beta}]$ *has zero measure if the following series diverges, and full measure if it converges:*

$$\sum_{l=1}^{\infty} \frac{1}{l \cdot \prod_{j=1}^{m} \beta_j(l) \cdot \prod_{i=1}^{n} \alpha_i(g(l))}$$

Khintchine–Groshev's theorem

Theorem (Khintchine–Groshev)

Let $f(t),\ t \geq 1$ be a non-increasing real valued function. The set $\mathbf{W}_{n,m}[f]$ has zero measure if the following series converges, and full measure if it diverges:

$$\sum_{k=1}^{\infty} f(k)$$

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Theorem (Khintchine–Groshev)

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$$\sum_{k=1}^{\infty} f(k)$$

Theorem (Kleinbock, Wang, 2023)

Let $f(T),\ T \geq 1$ be a non-increasing real valued function. The set $\mathbf{W}_{m,n}[f;\underline{\beta},\underline{\alpha}]$ has zero measure if the following series converges, and full measure if it diverges:

$$\sum_{k=1}^{\infty} k^{-1} \prod_{j=1}^{m} \beta_j(f(k)) \cdot \prod_{i=1}^{n} \alpha_i(k)$$

Proof of equivalence

- As usual,
$$g(T) = \frac{1}{f^{-1}\left(\frac{1}{T}\right)}, \qquad f(T) = \frac{1}{g^{-1}\left(\frac{1}{T}\right)}.$$
- $$\int\limits_1^{+\infty} \frac{dz}{z^2 g(z)} \qquad \text{and} \qquad \int\limits_1^{+\infty} f(t)dt$$
either both converge or both diverge.

Proof of equivalence: convergent case

- Suppose $\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)} < \infty$. By Khintchine-Groshev's theorem then for almost every Θ,

$$\Theta^\top \notin \mathbf{W}_{n,m}[f];$$

and thus

$$\widehat{\mathbf{D}}_{n,m}^{\Theta}\left[g \circ \frac{1}{\kappa^m}\right] = \mathbb{R}^n.$$

So almost every pair is $g \circ \frac{1}{\kappa^m}$-Dirichlet.

Proof of equivalence: divergent case

- Suppose $\sum\limits_{j=1}^{\infty} \frac{1}{j^2 g(j)} = \infty$. By Khintchine-Groshev's theorem then for almost every Θ,

$$\Theta^\top \in \mathbf{W}_{n,m}\left[f\right];$$

and so

$$\widehat{\mathbf{D}}_{n,m}^{\Theta}\left[\varepsilon \cdot g \circ \frac{1}{\varepsilon}\right]$$

has zero measure. Therefore almost no pairs are $\varepsilon \cdot g \circ \frac{1}{\varepsilon}$-Dirichlet.

What happens when measure is zero

We will complement our metric result by a statement of the following type: *when the function g decreases so fast that the measure of the sets $\widehat{\mathbf{D}}_{n,m}\left[g; \underline{\alpha}, \underline{\beta}\right]$ is zero, these sets are still nonempty and big in certain sense.*

Theorem (Hong, N. - work in progress)

For any decreasing function g, the set

$$\widehat{\mathbf{D}}_{n,m}\left[g; \underline{\alpha}, \underline{\beta}\right]$$

is uncountable and dense in $\mathbf{M}_{n,m} \times \mathbb{R}^n$.

On a criterion of bad approximability

Badly approximable matrices

Badly approximable matrices are those for which the asymptotic Dirichlet's theorem cannot be improved more than by a multiplicative constant:

$$\mathbf{BA}_{n,m} = \{\Theta \in \mathbf{M}_{n,m} : \ ||\Theta \mathbf{q}||^n_{\mathbb{Z}^n} > \frac{c}{|\mathbf{q}|^m} \text{ for some } c > 0 \text{ and any } \mathbf{q} \in \mathbb{Z}^m\}.$$

Despite being of measure zero, badly approximable matrices form a big set in the following sense:

Theorem (Schmidt +)

The set $\mathbf{BA}_{n,m}$ has Lebesgue measure zero. This set is winning in certain games ((α, β)-game, hyperplane absolute game...). In particular, this set is dense and has full Hausdorff dimension.

Badly approximable matrices: weight functions

Basing on Dirichlet's theorem for weight functions, we introduce the following natural definition:

Definition

Matrix $\Theta \in \mathbf{M}_{n,m}$ is called $(\underline{\alpha}, \underline{\beta})$-*badly approximable*, $\mathbf{BA}_{n,m}[\underline{\alpha}, \underline{\beta}]$, if it is not $(b \cdot g_{\underline{\alpha},\underline{\beta}}, \underline{\alpha}, \underline{\beta})$-approximable for some constant $b > 0$.

If α_i, β_j are quasimultiplicative, then by generalized Khintchine-Groshev's theorem the set $\mathbf{BA}_{n,m}[\underline{\alpha}, \underline{\beta}]$ is a Lebesgue nullset.

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If α_i, β_j are quasimultiplicative, then by generalized Khintchine-Groshev's theorem the set $\mathbf{BA}_{n,m}[\underline{\alpha}, \underline{\beta}]$ is a Lebesgue nullset.

Proposition

Let $\underline{\alpha}$ and $\underline{\beta}$ be tuples of quasimultiplicative functions. Suppose $\Theta \in \mathbf{M}_{n,m}$ is $(\underline{\alpha}, \underline{\beta})$-badly approximable. Then, $\Theta^\top$ is $(\underline{\beta}, \underline{\alpha})$-badly approximable.

Criterion of bad approximability

Corollary

Let $\underline{\alpha}$ and $\underline{\beta}$ be tuples of quasimultiplicative functions.

- *If a matrix $\Theta \in \mathbf{M}_{n,m}$ is $(\underline{\alpha}, \underline{\beta})$-badly approximable, then there exists $k > 0$ such that any pair $(\Theta, \boldsymbol{\eta})$ is $(kg_{\underline{\alpha},\underline{\beta}}, \underline{\alpha}, \underline{\beta})$-Dirichlet:*

$$\widehat{\mathbf{D}}^{\Theta}[kg_{\underline{\alpha},\underline{\beta}}, \underline{\alpha}, \underline{\beta}] = \mathbb{R}^n.$$

- *If a matrix $\Theta \in \mathbf{M}_{n,m}$ is not $(\underline{\alpha}, \underline{\beta})$-badly approximable, then for any $k > 0$ almost any pair $(\Theta, \boldsymbol{\eta})$ is not $(kg_{\underline{\alpha},\underline{\beta}}, \underline{\alpha}, \underline{\beta})$-Dirichlet:*

$$m\left(\widehat{\mathbf{D}}^{\Theta}[kg_{\underline{\alpha},\underline{\beta}}, \underline{\alpha}, \underline{\beta}]\right) = 0.$$

Criterion of bad approximability

Corollary

- *If a matrix* $\Theta \in \mathbf{M}_{n,m}$ *is badly approximable, then there exists* $k > 0$ *such that any pair* $(\Theta, \boldsymbol{\eta})$ *is* kf_1*-Dirichlet:*

$$\widehat{\mathbf{D}}^{\Theta}[kf_1] = \mathbb{R}^n.$$

- *If a matrix* $\Theta \in \mathbf{M}_{n,m}$ *is not* $(\underline{\alpha}, \underline{\beta})$*-badly approximable, then for any* $k > 0$ *almost any pair* $(\Theta, \boldsymbol{\eta})$ *is not* kf_1*-Dirichlet:*

$$m\left(\widehat{\mathbf{D}}^{\Theta}[kf_1]\right) = 0.$$

Criterion of bad approximability

Corollary

- *If a matrix* $\Theta \in \mathbf{M}_{n,m}$ *is badly approximable, then there exists* $k > 0$ *such that any pair* $(\Theta, \boldsymbol{\eta})$ *is* kf_1*-Dirichlet:*

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- *If a matrix* $\Theta \in \mathbf{M}_{n,m}$ *is not* $(\underline{\alpha}, \underline{\beta})$*-badly approximable, then for any* $k > 0$ *almost any pair* $(\Theta, \boldsymbol{\eta})$ *is not* kf_1*-Dirichlet:*

$$m\left(\widehat{\mathbf{D}}^{\Theta}[kf_1]\right) = 0.$$

In addition, the complement of $\widehat{\mathbf{D}}^{\Theta}[kf_1]$ *is a HAW set (winning in certain games; in particular Schmidt's game).*

Very good approximability

Let ρ and σ be weights.

Definition

Matrix $\Theta \in \mathbf{M}_{n,m}$ is called (ρ, σ)-*very well approximable* if it is $(f_{1+\varepsilon}; \rho, \sigma)$-approximable for some $\varepsilon > 0$.

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Theorem

- *If a matrix $\Theta \in \mathbf{M}_{n,m}$ is (ρ, σ)-very well approximable, then there exists $\varepsilon > 0$ such that almost no pair $(\Theta, \boldsymbol{\eta})$ is $[f_{1-\varepsilon}; \rho, \sigma]$-Dirichlet:*

$$m\left(\widehat{\mathbf{D}}^{\Theta}[f_{1-\varepsilon}; \rho, \sigma]\right) = 0.$$

- *If Θ is not (ρ, σ)-very well approximable, then for any $\varepsilon > 0$ and any $\boldsymbol{\eta} \in \mathbb{R}^n$ the pair $(\Theta, \boldsymbol{\eta})$ is $[f_{1-\varepsilon}; \rho, \sigma]$-Dirichlet:*

$$\widehat{\mathbf{D}}^{\Theta}[f_{1-\varepsilon}; \rho, \sigma] = \mathbb{R}^n.$$

Proof of "twisted Davenport-Schmidt's" theorem

Setup

By assumption, $\Theta^T \in \mathbf{W}_{m,n}[f; \underline{\beta}, \underline{\alpha}]$, which means that there exists a sequence $\mathbf{y}_\nu$ of vectors in $\mathbb{Z}^n$ for which

$$||\Theta^T \mathbf{y}_\nu||_{\underline{\beta}} \leq f(|\mathbf{y}_\nu|_{\underline{\alpha}}); \tag{3}$$

we will use the notation $Y_\nu = |\mathbf{y}_\nu|_\alpha$.

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we will use the notation $Y_\nu = |\mathbf{y}_\nu|_\alpha$.
By quasimuiltiplicativity, for any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that

$$\alpha_i(T)\alpha_i\left(\frac{\varepsilon}{T}\right) \leq \delta \qquad \text{and} \qquad \beta_j(\varepsilon T)\beta_j\left(\frac{1}{T}\right) \leq \delta \qquad \text{for any } i,j \text{ and any } T \in \mathbb{R}_+. \tag{4}$$

Let $\tilde{g} = \tilde{g}_{\varepsilon(\delta)} = \varepsilon \cdot g \circ \frac{1}{\varepsilon}$.

Outline of the proof

The proof of "twisted Davenport-Schmidt's" theorem consists of two parts:

Part 1. Use transference principle to show that the set

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}[\tilde{g};\underline{\alpha},\underline{\beta}]$$

is contained in a certain liminf set of a sequence $\{\Omega_\nu\}$ of sets which is explicitly described via $\{\mathbf{y}_\nu\}$.

Part 2. Show: if δ is small enough, the sets Ω_ν are "almost independent" and deduce an upper bound of $m\left(\liminf_{\nu\to\infty}\Omega_\nu\right)$.

Part 1: transference

We define

$$\Omega_\nu = \Omega_\nu\left((m+n)\delta\right) := \{\boldsymbol{\eta} \in [0,1)^n : \ \ ||\boldsymbol{\eta}\cdot \mathbf{y}_\nu|| \leq (m+n)\delta\}.$$

Lemma

Let $\delta > 0$ be a parameter, and $\mathbf{y}_\nu, \tilde{g}$ and Ω_ν be chosen as above. Then,

$$\widehat{\mathbf{D}}^{\Theta}_{n,m}[\tilde{g}; \underline{\alpha}, \underline{\beta}] \cap [0,1)^n \subseteq \bigcup_{N=1}^{\infty} \bigcap_{\nu=N}^{\infty} \Omega_\nu.$$

Part 1: transference

Suppose

$$\boldsymbol{\eta} \in \widehat{\mathbf{D}}^{\Theta}_{n,m}[\tilde{g}; \underline{\alpha}, \underline{\beta}] \cap [0,1)^n,$$

and let $T_\nu = \frac{\varepsilon}{f(Y_\nu)}$; for ν large enough there exists $\mathbf{q}_\nu$ such that

$$\begin{cases} |q_{\nu,j}| \leq \beta_j(T_\nu), & j = 1, \ldots, m; \\ ||\Theta_i \mathbf{q}_\nu - \eta_i|| \leq \alpha_i\left(\tilde{g}(T_\nu)\right), & i = 1, \ldots, n. \end{cases}$$

Part 1: transference

Classical transference observation:

$$\boldsymbol{\eta} \cdot \mathbf{y}_\nu = \sum_{j=1}^{m} q_j \Theta_j^T(\mathbf{y}_\nu) - \sum_{i=1}^{n} (\Theta_i(\mathbf{q}) - \eta_i)\, y_{\nu,i}.$$

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Bound both terms on the right hand side:

$$||q_j \Theta_j^T(\mathbf{y}_\nu)|| \leq |q_i| \cdot ||\Theta_j^T(\mathbf{y}_\nu)|| \leq \beta_j(T_\nu)\beta_j(f(Y_\nu)) = \beta_j\left(\frac{\varepsilon}{f(Y_\nu)}\right)\beta_j(f(Y_\nu))$$

and

$$||\,(\Theta_i(\mathbf{q}) - \eta_i)\, y_{\nu,i}|| \leq |y_{\nu,i}| \cdot ||\,(\Theta_i(\mathbf{q}) - \eta_i)\,|| \leq \alpha_i(Y_\nu)\alpha_i(\tilde{g}(T_\nu)) = \alpha_i(Y_\nu)\alpha_i\left(\frac{\varepsilon}{Y_\nu}\right).$$

Part 1: transference

Combine these estimates: by the choice of ε we get that

$$||\boldsymbol{\eta} \cdot \mathbf{y}_\nu|| \leq \sum_{j=1}^{m} ||q_j \Theta_j^T(\mathbf{y}_\nu)|| + \sum_{i=1}^{n} ||\left(\Theta_i(\mathbf{q}) - \eta_i\right) y_{\nu,i}|| \leq (m+n)\delta.$$

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The condition above is $\boldsymbol{\eta} \in \Omega_\nu$ and it should hold for all ν large enough, so

$$\boldsymbol{\eta} \in \bigcup_{N=1}^{\infty} \bigcap_{\nu=N}^{\infty} \Omega_\nu.$$

Part 2: measure estimate

Lemma

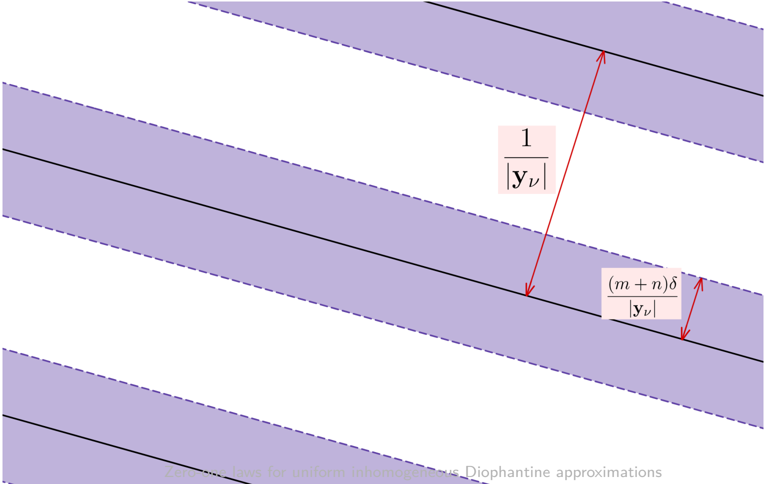
Let $0 < \delta < \frac{1}{2(n+m)}$. *Then,*

$$m\left(\bigcup_{N=1}^{\infty}\bigcap_{\nu=N}^{\infty}\Omega_\nu\right) = 0.$$

First, we want to switch from $\{\mathbf{y}_\nu\}$ to a subsequence with an additional condition $\lim\limits_{\nu\to\infty}\frac{|\mathbf{y}_{\nu+1}|}{|\mathbf{y}_\nu|} = \infty$. It can only increase the liminf set we are working with.

Part 2: measure estimate

The sets $\Omega_\nu = \{\boldsymbol{\eta} \in [0,1)^n : \ ||\boldsymbol{\eta} \cdot \mathbf{y}_\nu|| \le (m+n)\delta\}$ consist of "stripes" — $\frac{(m+n)\delta}{|\mathbf{y}_\nu|}$-neighborhoods of the hyperplanes $\boldsymbol{\eta} \cdot \mathbf{y}_\nu = p$ where $p \in \mathbb{Z}$:



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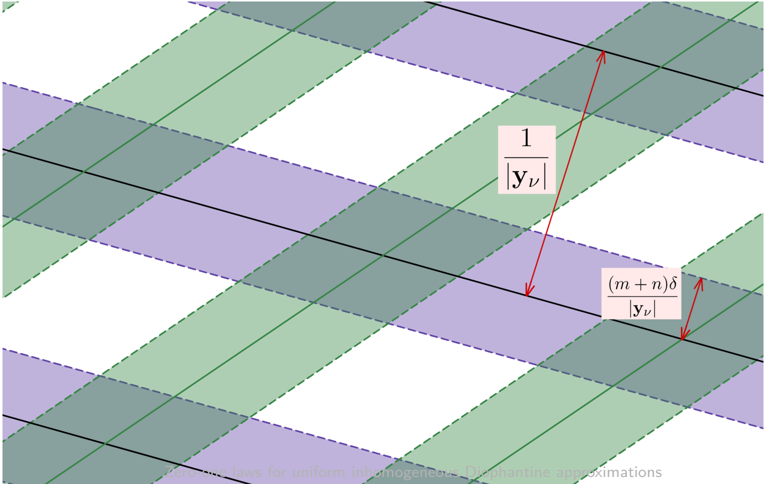

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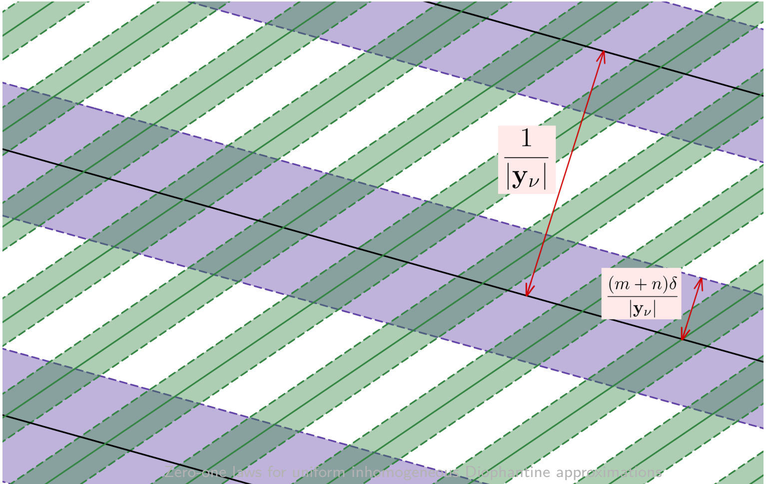

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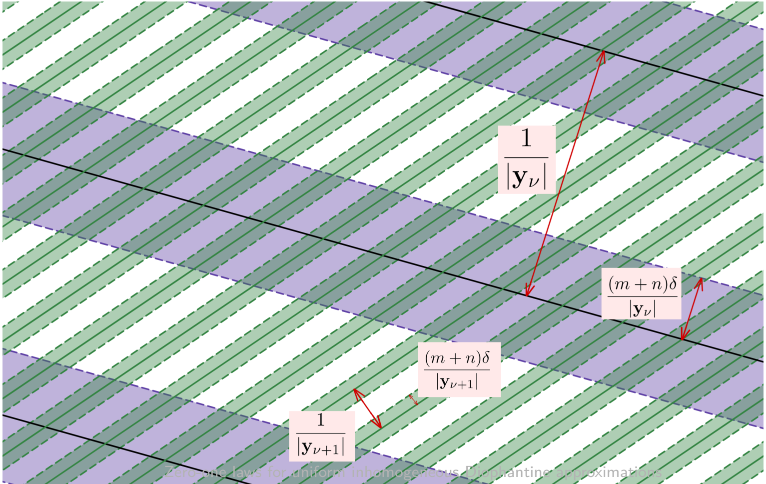

Part 2: measure estimate

Claim. For $\nu \neq \mu$ large enough,

$$m\left(\Omega_\nu^c \cap \Omega_\mu^c\right) \leq (1+\lambda) m\left(\Omega_\nu^c\right) m\left(\Omega_\mu^c\right).$$

Part 2: measure estimate

Claim. For $\nu \neq \mu$ large enough,

$$m\left(\Omega_\nu^c \cap \Omega_\mu^c\right) \leq (1+\lambda) m\left(\Omega_\nu^c\right) m\left(\Omega_\mu^c\right).$$

This allows us to apply the following classical statement to $\limsup\limits_{\nu\to\infty} \Omega_\nu^c$:

Lemma

Let X be a measure space with measure μ, and let $\{E_k\}_{k=1}^\infty$ be a countable collection of measurable sets. Let $E = \bigcap\limits_{N=1}^\infty \bigcup\limits_{k=N}^\infty E_k$. Suppose $\sum\limits_{k=1}^\infty \mu(E_k) = \infty$. Then,

$$\mu(E) \geq \limsup_{N\to\infty} \frac{\left(\sum\limits_{k=1}^N \mu(E_k)\right)^2}{\sum\limits_{k,s=1}^N \mu(E_k \cap E_s)}.$$

Exceptional subspaces

Ξ-Exceptional subspaces

Let $\{\mathbf{y}_\nu\}$ be the sequence used in the proof of "inhomogeneous Davenport-Schmidt's" theorem.

Definition

We call a point $\mathbf{v} \in \mathbb{R}^n$ on the unit sphere (direction) Ξ-*exceptional* (for the matrix $\Theta^\top$ and function f) if for any $\delta > 0$ there exist infinitely many $\nu \in \mathbb{N}$ for which $\widehat{\mathbf{y}_\nu, \mathbf{v}} < \frac{\delta}{|\mathbf{y}_\nu|^\Xi}$.

Definition

Let $\mathcal{A}$ be an affine subspace of $\mathbb{R}^n$ and $\mathcal{L}$ — the corresponding linear subspace. We call $\mathcal{A}$ Ξ-*exceptional* if $\mathcal{L}^\perp$ contains Ξ-exceptional points.

Ξ-Exceptional subspaces

Informally, non-exceptional subspaces are those where the projections of $\mathbf{y}_\nu$ grow fast enough.

Lemma

Let $\Xi > 0$ and X_E be the set of k-dimensional Ξ-exceptional linear subspaces of $\mathbb{R}^n$.
Then, $\dim_H X_E = k(n-k-1)$, if not empty.
In particular, almost no affine subspaces are Ξ-exceptional.

The "inhomogeneous Davenport-Schmidt theorem" works for $\boldsymbol{\eta}$ restricted on any non-exceptional subspace of $\mathbb{R}^n$.

Thank you for your attention!