Badly approximability, Schmidt's games and completely irrational subspaces

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Let $0 < \alpha, \beta < 1$. Suppose that two players Bob and Alice choose in turn a nested sequence of closed balls in $\mathbb{R}^d$:

$$B_1 \supset A_1 \supset B_2 \supset \ldots$$

with the property

$$\rho(A_i) = \alpha\rho(B_i),\ \ \rho(B_{i+1}) = \beta\rho(A_i)\ \text{ for all } i = 1, 2, \ldots .$$

A set $E \subset \mathbb{R}^d$ is called (α, β)-*winning* if Alice has a strategy guaranteeing that the intersection

$$\bigcap_{i=0}^{\infty} A_i$$

belongs to E no matter how Bob plays.
A set $E \supset \mathbb{R}^d$ is called α-*winning* if it is (α, β)-winning for all $0 < \beta < 1$.

Lemma

If $2\beta < 1 + \alpha\beta$, then every (α, β)-winning set has the cardinality of the continuum. For $\alpha > 0$ any α-winning set has full Hausdorff dimension.

Lemma

The intersection of countably many α-winning sets is α-winning.

Lemma

Suppose that $0 < \alpha, \beta < 1$ *and* $\gamma = 1 + \alpha\beta - 2\beta > 0$*. Let* t *be an integer with* $(\alpha\beta)^t < \frac{1}{2}\gamma$ *and* u *a vector of length 1. Suppose a ball* A_k *occurs in the* (α, β)*-game. Then Bob can play so that (no matter how Alice plays) every point* x *of* A_{k+t} *satisfies*

$$((x - o_k), u) > \frac{1}{2}\gamma\rho(A_k)$$

where o_k *denotes the center of* A_k*.*

Fix $\beta < \frac{1}{3}$. Firstly, Bob chooses a closed ball $B_1 = B_1(x_1, \rho_1)$ with center x_1 and radius ρ_1 in $\mathbb{R}^d$. Then in each stage of the game, after Bob chooses B_i, Alice chooses an affine subspace L of dimension $d - 1$ and removes its ε-neighbourhood A_i from B_i, where $0 < \varepsilon < \beta\rho_i$ (and ε can be different depending on i).

Then Bob chooses the next ball B_{i+1} with radius $\rho_{i+1} \geq \beta\rho_i$ and under condition

$$B_{i+1} \subset B_i \setminus A_i.$$

The set S is said to be *hyperplane β-absolute winning* if Alice has a strategy guaranteeing that

$$\bigcap_{i=0}^{\infty} B_i$$

intersects S.

We say S is *hyperplane absolute winning* if it is *hyperplane β-absolute winning* for every $0 < \beta < \frac{1}{3}$.

Properties of HAW sets

- The countable intersection of hyperplane absolute winning sets is hyperplane absolute winning.
- HAW implies α-winning for $\alpha < \frac{1}{2}$.

Theorem

(Dirichlet, 1842) Let α be a real number. Then for any real number t there exist such an integer p and $q \in \mathbb{N}$ that

$$|\alpha - \frac{p}{q}| < \frac{1}{tq}.$$

Corollary

Let α be an irrational real number. Then there exist infinitely many pairs (p, q) of integers such that

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$

Definition

A number α is called badly approximable if there exists such a constant $c > 0$ that for all integers p, q one has

$$|\alpha - \frac{p}{q}| > \frac{c}{q^2}.$$

Let

$$L_j(x) = \theta_j^1 x_1 + \ldots + \theta_j^n x_n, \ \ j = 1, \ldots, m$$

be a system of linear forms with real coefficients.
Notation. By $||\cdot||$ we denote the distance from the nearest integer. That is, $||\alpha|| = \min\limits_{a \in \mathbb{Z}} |\alpha - a|$.

Theorem

There exists infinitely many integer points $x \in \mathbb{Z}^n$ such that

$$(\max(|x_1|, \ldots, |x_n|))^n (\max(\| L_1(x) \|, \ldots, \| L_m(x) \|))^m < 1.$$

Let

$$L_j(x) = \theta_j^1 x_1 + \dots + \theta_j^n x_n, \;\; j = 1, \dots, m$$

be a system of linear forms with real coefficients and suppose there exists a constant $c > 0$ such that for all $x \in \mathbb{Z}^n \setminus \{0\}$

$$(\max(|x_1|, \dots, |x_n|))^n (\max(\| L_1(x) \|, \dots, \| L_m(x) \|))^m > c.$$

Then this system is called badly approximable.
We denote the set of matrices $\Theta = (\theta_j^i)$ of badly approximable linear forms by $N(n, m)$.
We call n-dimensional linear subspace $\mathcal{L}$ in $\mathbb{R}^d$ *badly approximable* if

$$\inf_{x \in \mathbb{Z}^n \setminus \{0\}} (\max(|x_1|, \dots, |x_n|))^n \mathrm{dist}(\mathcal{L}, x)^m > 0$$

Theorem

(Schmidt, 1969.) The set $N(n, m)$ is (α, β)-winning if

$$2\alpha < 1 + \alpha\beta.$$

The following result was proved by Broderick, Fishman, Simmons:

Theorem

The set $N(n, m)$ is HAW.

Definition

We define n-dimensional subspace $\mathcal{L}$ in $\mathbb{R}^d$ to be *completely irrational* if its intersection with any $d-n$-dimensional rational subspace is just $\{0\}$.

A simple sufficient condition:
If Plücker coordinates of the subspace $\mathcal{L} \subset \mathbb{R}^d$ are lineary independent over $\mathbb{Q}$, then $\mathcal{L}$ is completely irrational.
We denote the set of all $n \times m$-matrices Θ defining n-dimensional completely irrational subspace of the form

$$\mathcal{L} = \{(x, y) : \ \ x \in \mathbb{R}^n, \ \ y = \Theta x\}$$

in $\mathbb{R}^{n+m}$ by $I(n, m)$.

Theorem

Suppose that $2\alpha < 1 + \alpha\beta$. Then the set $I(n, m)$ is (α, β)-winning. In particular $I(n, m)$ is α-winning for $\alpha \leq \frac{1}{2}$.

Theorem

The set $N^{irr}(n, m)$ of $n \times (d - n)$-matrices defining badly approximable n-dimensional completely irrational subspaces is α-winning if $\alpha \leq \frac{1}{2}$.

Theorem

For any $n, m \in \mathbb{N}$ the following statements hold.

1. *The set $I(n, m)$ is HAW, and so*
2. *The set $N^{irr}(n, m)$ is HAW.*

Lemma

Consider (α,β)-game in $\mathbb{R}^r$, and suppose $0<\alpha<1, 0<\beta<1, \gamma=1+\alpha\beta-2\beta>0$. Let $f\in\mathbb{R}[z_1,...,z_r]$ be a nonzero polynomial and $\{f(z)=0\}$ be the corresponding algebraic manifold M. Suppose a ball A_{l_0} occurs in the (α,β)-game. Then Bob can play so that (no matter how Alice plays) for some $\epsilon>0$ and for some $l\geq l_0$ any point $y\in A_l$ satisfies the inequality

$$\mathrm{dist}(y,M)>\varepsilon.$$

The same statement holds also for HAW instead of Schmidt's (α,β)-game.

Applications: Dimension of approximations

Let $L_j(x)$ for $j = 1, ..., m$ be a system of linear forms with real coefficients with matrix Θ, as before. For an integral vector $x \in \mathbb{Z}^n$ let's consider the quantity $\zeta(x) = \max\limits_{j=1,\ldots,m} ||L_j(x)||$.

An integer vector $x = (x_1, ..., x_n)$ is called a best approximation for matrix Θ if

$$\zeta(x) = \min_{x'} \zeta(x')$$

where minimum is taken over all $x' \in \mathbb{Z}^n \setminus \{0\}$ s.t.

$$\max_{i=1,\ldots,n} |x'|_i \leq \max_{i=1,\ldots,n} |x|_i.$$

A $n \times m$ matrix Θ is called *good* if the sequence of its *best approximations* $\{x_\nu\}_{\nu=1}^{\infty}$ is unique and well defined.

For a good matrix Θ (or subspace $\mathcal{L}$) we define

$$R(\Theta) = R(\mathcal{L}) = \min\{r : \text{ there exists a linear subspace } \mathcal{B} \subseteq \mathbb{R}^d, \mathrm{dim}\mathcal{B} = r,$$

$$\text{and } \nu_0 \in \mathbb{N} \text{ such that } z_\nu \in \mathcal{B} \text{ for all } \nu \geq \nu_0\}.$$

If $\mathcal{L}$ is completely irrational, the following statement holds.

Theorem

Suppose $\mathcal{L}$ is an n-dimensional good completely irrational subspace in $\mathbb{R}^d$ of form $y = \Theta x$. Then

$$R(\Theta) = R(\mathcal{L}) \geq d - n + 1.$$

The following statement is due to Nikolay Moshchevitin.

Theorem

Suppose 2×2*-matrix* Θ *of two-dimensional subspace* $\mathcal{L}$ *in* $\mathbb{R}^4$ *is good and* $\mathcal{L}$ *is not contained in a rational 3-dimensional subspace of* $\mathbb{R}^4$*. Then* $R(\Theta) = 2$ *or* $R(\Theta) = 4$.

As completely irrational two-dimensional subspace in $\mathbb{R}^4$ can not lie in any 3-dimensional rational subspace, we immediately deduce

Corollary

Suppose 2×2*-matrix* Θ *of two-dimensional completely irrational subspace* $\mathcal{L}$ *in* $\mathbb{R}^4$ *is good. Then* $R(\Theta) = 4$.

We consider vectors ξ is of the form

$$\xi = (\xi_1, \ldots, \xi_{d-1}, 1).$$

The vector ξ is called *totally irrational* if its coordinates are linearly independent over $\mathbb{Q}$.
The ordinary Diophantine exponent $\omega(\xi)$ of ξ is the supremum of the set of all $\gamma > 0$ for which the inequality

$$\max_{j=1,\ldots,d-1} \| q\xi_j \| \leq q^{-\gamma}$$

has infinitely many integer solutions $q > 0$.
The uniform Diophantine exponent $\hat{\omega}(\xi)$ of ξ is the supremum of the set of all $\gamma > 0$ for which the system of inequalities

$$\max_{j=1,\ldots,d-1} \| q\xi_j \| \leq t^{-\gamma}, \quad 0 < q \leq t,$$

has an integer solution q for all t large enough.
For irrational ξ we have $\frac{1}{d-1} \leq \hat{\omega}(\xi) \leq 1$ and $\omega(\xi) \geq \hat{\omega}(\xi)$.

Let

$$w_{n,d} = \frac{n}{d-n}$$

and let $W_{n,d}$ be the unique root of the equation

$$x^d - w_{n,d}^{d-2}(1 + w_{n,d})x + w_{n,d}^{d-1} = 0$$

in the interval $(0, w_{n,d})$.

Theorem

(Kleinbock, Moshchevitin, Weiss). Let $\mathcal{L}$ be an $n-$ dimensional badly approximable linear subspace of $\mathbb{R}^d$. Then

1. *for any $\xi \in \mathcal{L}$ one has*
$$\hat{\omega}(\xi) \leq w_{n,d};$$
2. *for any totally irrational $\xi \in \mathcal{L}$ one has*
$$\hat{\omega}(\xi) \leq W_{n,d}.$$

An improvement for a completely irrational subspaces

Let $\mathfrak{W}_{n,d}$ be the unique root of the equation

$$x^{d-n+1} - w_{n,d}^{d-n-1}(1 + w_{n,d})x + w_{n,d}^{d-n} = 0$$

in the interval $(0, w_{n,d})$.

Theorem

Suppose $\mathcal{L}$ is an $n-$ dimensional badly approximable completely irrational linear subspace of $\mathbb{R}^d$. Then for any $\xi \in \mathcal{L}$ one has

$$\hat{\omega}(\xi) \leq \mathfrak{W}_{n,d}.$$

Let $\mathcal{L}$ be a 2-dimensional badly approximable linear subspace in $\mathbb{R}^4$. Then:

- For any $\xi \in \mathcal{L}$ we have

$$\hat{\omega}(\xi) \leq w_{2,4} = 1$$

- For any totally irrational $\xi \in \mathcal{L}$ we have

$$\hat{\omega}(\xi) \leq W_{2,4} \approx 0,54\ldots$$

- If $\mathcal{L}$ is completely irrational, for any $\xi \in \mathcal{L}$ we have

$$\hat{\omega}(\xi) \leq \mathfrak{W}_{2,4} = \frac{\sqrt{5}-1}{2} \approx 0,62\ldots$$

Thank you for your attention!