Inhomogeneous Diophantine approximations: some recent advances

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Introduction

Notation

We will use the following conventions:

- m and n are positive integers and $d = m + n$. $\mathrm{Mat}_{m,n}$ is the set of all real $m \times n$ matrices.
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$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \cdots & \cdots & \cdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \in \mathrm{Mat}_{m,n}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$$
  — a pair of a real $m \times n$ matrix and m-vector.
- $|\cdot|$ is the supremum norm of a vector, and $|\cdot|_{\mathbb{Z}^m}$ is the distance to the nearest integer vector.

Notation

- Classical homogeneous Diophantine approximations studies the form

$$|A\mathbf{q}|_{\mathbb{Z}^m}$$

- In the inhomogeneous setting, we replace the linear for with affine: we study

$$|A\mathbf{q} - \mathbf{b}|_{\mathbb{Z}^m}$$

More specifically, we want to compare the values of this forms with values of certain functions:

Uniform and asymptotic approximations

Fix a non-increasing real function g. We study the system of inequalities

$$\begin{cases} |A\mathbf{q}|_{\mathbb{Z}^m}^m \le g(T) \\ |\mathbf{q}|^n \le T \end{cases} \tag{1}$$

In both homogeneous and inhomogeneous setups, two types of questions are of interest:

- Uniform approximations: does the system (1) have an integer solution $\mathbf{q}$ for all T large enough? $\mathbf{D}_{m,n}[g]$ – g-Dirichlet matrices
- Asymptotic approximations: does the system (1) have an integer solution $\mathbf{q}$ for an unbounded set of T? $\mathbf{W}_{m,n}[g]$ – g-approximable matrices

Uniform and asymptotic approximations

Fix a non-increasing real function g. We study the system of inequalities

$$\begin{cases} |A\mathbf{q} - \mathbf{b}|_{\mathbb{Z}^m}^m \leq g(T) \\ |\mathbf{q}|^n \leq T \end{cases} \tag{2}$$

In both homogeneous and inhomogeneous setups, two types of questions are of interest:

- Uniform approximations: does the system (2) have an integer solution $\mathbf{q}$ for all T large enough? $\widehat{\mathbf{D}}_{m,n}[g]$ – g-Dirichlet pairs
- Asymptotic approximations: does the system (2) have an integer solution $\mathbf{q}$ for an unbounded set of T? $\widehat{\mathbf{W}}_{m,n}[g]$ – g-approximable pairs

Sections of sets of pairs

In the inhomogeneous case we can study three types of objects:

- Sets of pairs $(A, \mathbf{b})$, satisfying some restrictions: $\widehat{\mathbf{D}}_{m,n}[g]$, $\widehat{\mathbf{W}}_{m,n}[g]$.
- Sets of matrices A for a fixed "shift vector" $\mathbf{b}$ ("sections with a fixed vector"): $\widehat{\mathbf{D}}^{\mathbf{b}}_{m,n}[g]$, $\widehat{\mathbf{W}}^{\mathbf{b}}_{m,n}[g]$.
- Sets of vectors $\mathbf{b}$ for a fixed matrix A ("Twisted approximations", or "sections with a fixed matrix"): $\widehat{\mathbf{D}}^{A}_{m,n}[g]$, $\widehat{\mathbf{W}}^{A}_{m,n}[g]$.

Function g is always a non-increasing real valued function.

Overview of results

Homogeneous uniform approximation

We will use the notation: $f_a(T) = \frac{1}{T^a}$, in particular, $f_1(T) = \frac{1}{T}$.

Theorem (Dirichlet)

Any matrix $A \in \mathrm{Mat}_{m,n}$ is f_1-Dirichlet:

$$\mathbf{D}_{m,n}\left[f_1\right] = \mathrm{Mat}_{m,n}.$$

Theorem (Dirichlet improvable matrices have measure zero; Davenport, Schmidt)

If $\varepsilon > 0$ is small enough (in fact < 1), the set of ε-Dirichlet improvable matrices has Lebesgue measure zero:

$$\mathcal{L}^{mn}\left(\mathbf{D}_{m,n}\left[\varepsilon f_1\right]\right) = 0.$$

Homogeneous asymptotic approximation

Theorem (Dirichlet)

Any matrix $A \in \mathrm{Mat}_{m,n}$ *is* f_1*-approximable:*

$$\mathbf{W}_{m,n}\left[f_1\right] = \mathrm{Mat}_{m,n}.$$

Theorem (Khintchine-Groshev – zero-one law)

The set $\mathbf{W}_{m,n}\left[f\right]$ *has zero measure if the series*

$$\sum_{k=1}^{\infty} f(k)$$

converges, and full measure if it diverges.

Transference principle

From now on, positive decreasing to zero functions f and g will always be related by

$$g(T) = \frac{1}{f^{-1}\left(\frac{1}{T}\right)}, \qquad f(T) = \frac{1}{g^{-1}\left(\frac{1}{T}\right)}.$$

Remark

The faster f decreases, the slower g decreases.

It is due to Jarnik that:

- The asymptotic approximations of the matrix $A^\top$ are related to uniform approximations of pairs of the form $(A, \mathbf{b})$.
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Geometry behind transference: lattices and grids

Put $d = m + n$ and

$$u_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}, \qquad \widetilde{\mathbf{b}} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix} \in \mathbb{R}^d.$$

The matrix and the pair determine, respectively, a lattice and a grid:

$$\Lambda_A := u_A \mathbb{Z}^d, \qquad \Lambda_{A,\mathbf{b}} := \Lambda_A - \widetilde{\mathbf{b}}, \qquad \Lambda_A^* := (u_A^{-1})^\top \mathbb{Z}^d.$$

For $U, T > 0$, let

$$a(U,T) := \operatorname{diag}\left(U^{-1/m} I_m, T^{-1/n} I_n\right), \qquad \mathcal{B}_d := [-1,1]^d.$$

Dani's correspondence gives

$$A^\top \in \mathbf{W}_{n,m}[f] \iff a(U, f(U))\Lambda_A^* \cap \mathcal{B}_d \neq \{\mathbf{0}\} \quad \text{for unbounded } U,$$
$$(A,\mathbf{b}) \in \widehat{\mathbf{D}}_{m,n}[g] \iff a(g(T), T)\Lambda_{A,\mathbf{b}} \cap \mathcal{B}_d \neq \varnothing \quad \text{for all } T \gg 1.$$

When $\mathbf{b} = \mathbf{0}$, the second intersection is required to contain a nonzero point.

Geometry behind transference: duality and covering

If $A^\top \notin \mathbf{W}_{n,m}[f]$, Dani's correspondence gives, for all sufficiently large U,

$$a(U, f(U))\Lambda_A^* \cap \mathcal{B}_d = \{\mathbf{0}\}.$$

Duality gives

$$\left[a\left(\frac{1}{U}, \frac{1}{f(U)}\right)\Lambda_A\right]^* = a(U, f(U))\Lambda_A^*.$$

Since $\mathcal{B}_d$ is self-pseudo-compound, Mahler's theorem yields C_d such that

$$(C_d\mathcal{B}_d + \boldsymbol{\gamma}) \cap a\left(\frac{1}{U}, \frac{1}{f(U)}\right)\Lambda_A \neq \varnothing \qquad \left(\boldsymbol{\gamma} \in \mathbb{R}^d,\ U \gg 1\right).$$

Choose the translate determined by $\widetilde{\mathbf{b}}$ and use $T = 1/f(U)$, $g(T) = 1/U$:

$$C_d\mathcal{B}_d \cap a(g(T), T)\Lambda_{A,\mathbf{b}} \neq \varnothing \qquad (T \gg 1) \quad \text{(with a nonzero point if } \mathbf{b} = \mathbf{0}\text{)}.$$

Only the first minimum. The only Diophantine input is shortest-vector avoidance for the dual trajectory. Mahler turns it into uniform covering of every shifted grid, up to a fixed rescaling.

New: Uniform twisted approximations via transference

Theorem (Jarnik+)

Suppose the transposed matrix $A^\top$ is not f-approximable, that is, $A^\top \notin \mathbf{W}_{n,m}[f]$. Then, any pair $(A, \mathbf{b})$ is $g \circ \frac{1}{\kappa^n}$-Dirichlet, where $\kappa = \kappa(m,n)$; that is,

$$\widehat{\mathbf{D}}^A_{m,n}\left[g \circ \frac{1}{\kappa^n}\right] = \mathbb{R}^m.$$

Theorem (Moshchevitin, N., 2025)

Suppose $A^\top$ is an f-approximable matrix. If ε is small enough, then the set

$$\widehat{\mathbf{D}}^A_{m,n}\left[\varepsilon \cdot g \circ \frac{1}{\varepsilon}\right]$$

has Lebesgue measure zero. Works also for intersections with certain subspaces.

New: Asymptotic twisted approximations via transference

Theorem (Jarnik+)

Suppose $A^\top$ is a non f-Dirichlet matrix. Then, for any $\mathbf{b} \in \mathbb{R}^m$ the pair $(A, \mathbf{b})$ is $g \circ \frac{1}{\kappa^n}$-approximable:

$$\widehat{\mathbf{W}}^{A}_{m,n}\left[g \circ \frac{1}{\kappa^n}\right] = \mathbb{R}^m.$$

The next theorem is a nonoptimal analogue of the convergence part of Khintchine–Groshev.

Theorem (Moshchevitin, N., 2025)

Suppose that $A^\top$ is an f-Dirichlet matrix. Let $\tilde{g}$ be an infinitely small function compared to g, defined in such a way that a certain series converges. Then the set $\widehat{\mathbf{W}}^{A}_{m,n}[\tilde{g}]$ has Lebesgue measure zero. Works also for intersections with certain subspaces.

Beyond transference: the full minima profile

Recall $\Lambda_A = u_A \mathbb{Z}^d$. The standard geodesic flow is

$$a_t = \begin{pmatrix} e^{t/m} I_m & 0 \\ 0 & e^{-t/n} I_n \end{pmatrix} = a(e^{-t}, e^t).$$

Let $\lambda_i(\Lambda)$ be the i-th successive minimum of the lattice Λ, and put

$$\lambda_i(t) := \lambda_i \left(a_t \Lambda_A \right), \qquad 1 \le i \le d.$$

- $\lambda_1(t)$ detects the existence of one short lattice vector; this is the information used in the transference argument.
- $\lambda_i(t)$ records when the stretched cube contains i linearly independent lattice vectors.
- Sharp coverings and fractal constructions must control clustering and the available independent directions, and therefore use the full profile $\lambda_1, \dots, \lambda_d$.

Beyond transference: the full minima profile

Theorem (Kim, N. – ongoing; asymptotic zero-one law.)

Assume that g is positive and non-increasing, and $T \mapsto Tg(T)$ is non-increasing. Then $\widehat{\mathbf{W}}^{A}_{m,n}[g]$ has zero or full Lebesgue measure, according as an explicit series involving $\lambda_1, \ldots, \lambda_d$ converges or diverges.

Theorem (Kim, N. – ongoing; Hausdorff dimensions in the uniform setup.)

Assume that g is quasimultiplicative: its decay under $T \mapsto RT$ is bounded between two powers of R. Two values determined by the full successive-minima profile give the Hausdorff dimensions of

$$\bigcap_{c>0} \widehat{\mathbf{D}}^{A}_{m,n}[cg] \qquad \textit{and} \qquad \bigcup_{c>0} \widehat{\mathbf{D}}^{A}_{m,n}[cg].$$

Remark

If the two values above coincide, we get the exact expression for $\dim_H \widehat{\mathbf{D}}^{A}_{m,n}[cg]$, which is c-independent. In some sense, this behavior is generic.

Dirichlet's theorem for pairs

Inhomogeneous Dirichlet's theorem

Theorem (Kleinbock and Wadleigh, 2019)

Let $g(z),\ z \geq 1$ be a non-increasing real valued function. The set $\widehat{\mathbf{D}}_{m,n}[g]$ has zero measure if the following series diverges, and full measure if it converges:

$$\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)}$$

Inhomogeneous Dirichlet's theorem

Theorem (Kleinbock and Wadleigh, 2019)

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$$\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)}$$

Theorem (Khintchine–Groshev)

Let $f(t)$, $t \geq 1$ be a non-increasing real valued function. The set $\mathbf{W}_{m,n}[f]$ has zero measure if the following series converges, and full measure if it diverges:

$$\sum_{k=1}^{\infty} f(k)$$

Proof of equivalence

- As usual,

$$g(T) = \frac{1}{f^{-1}\left(\frac{1}{T}\right)}, \qquad f(T) = \frac{1}{g^{-1}\left(\frac{1}{T}\right)}.$$

- $$\int_1^{+\infty} \frac{dz}{z^2 g(z)} \qquad \text{and} \qquad \int_1^{+\infty} f(t)dt$$

  either both converge or both diverge.

Proof of equivalence: convergent case

- Suppose $\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)} < \infty$. By Khintchine-Groshev's theorem then for almost every A,

$$A^\top \notin \mathbf{W}_{n,m}[f];$$

and thus

$$\widehat{\mathbf{D}}_{m,n}^{A}\left[g \circ \frac{1}{\kappa^n}\right] = \mathbb{R}^m.$$

So almost every pair is $g \circ \frac{1}{\kappa^n}$-Dirichlet.

Proof of equivalence: divergent case

- Suppose $\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)} = \infty$. By Khintchine-Groshev's theorem then for almost every A,

$$A^\top \in \mathbf{W}_{n,m}\,[f]\,;$$

and so

$$\widehat{\mathbf{D}}_{m,n}^{A}\left[\varepsilon \cdot g \circ \frac{1}{\varepsilon}\right]$$

has zero measure. Therefore almost no pairs are $\varepsilon \cdot g \circ \frac{1}{\varepsilon}$-Dirichlet.

Weighted case

$$\sum_{i=1}^{m} \rho_i = \sum_{j=1}^{n} \sigma_j = 1.$$

Define the quasinorms $|\cdot|_\rho$ and $|\cdot|_\sigma$ on $\mathbb{R}^m$ and $\mathbb{R}^n$ by

$$|\mathbf{y}|_\rho = \max_{i=1,\ldots,m} |y_i|^{\frac{1}{\rho_i}} \qquad \text{and} \qquad |\mathbf{x}|_\sigma = \max_{j=1,\ldots,n} |x_j|^{\frac{1}{\sigma_j}}.$$

Let $\widehat{\mathbf{D}}_{m,n}[g;\rho,\sigma]$ be the set of g-Dirichlet pairs with respect to weighted quasinorms.

Theorem (N., 2026)

Let $g(z),\ \ z \geq 1$ be a non-increasing real valued function. The set $\widehat{\mathbf{D}}_{m,n}[g;\rho,\sigma]$ has zero measure if the following series diverges, and full measure if it converges:

$$\sum_{j=1}^{\infty} \frac{1}{j^2 g(j)}$$

What happens when measure is zero

We call a matrix A *trivially singular* if there exists $\mathbf{q} \in \mathbb{Z}^n$ such that $A\mathbf{q} \in \mathbb{Z}^m$. Clearly, for any positive function g one has $A \in \mathbf{D}_{m,n}[g]$ (since $|A\mathbf{q}|_{\mathbb{Z}^m} = 0$). We call a pair $(A, \mathbf{b})$ *trivially singular* if there exists $\mathbf{q} \in \mathbb{Z}^n$ such that $A\mathbf{q} - \mathbf{b} \in \mathbb{Z}^m$.

Theorem (Khintchine, Jarnik)

Let $n \geq 2$. For any decreasing positive function g, the set

$$\mathbf{D}_{m,n}[g] \setminus \{\textit{trivially singular matrices}\}$$

is uncountable and dense in $\mathrm{Mat}_{m,n}$.

Proof idea: we rely on density + connectedness of the set

$$\bigcup_{\mathbf{q}\in\mathbb{Z}^n,\ \mathbf{p}\in\mathbb{Z}^m} \{A \in \mathrm{Mat}_{m,n} :\ A\mathbf{q} = \mathbf{p}\}$$

of trivially singular matrices.

What happens when measure is zero

Theorem (Khintchine + Moshchevitin, N. + Hong, Kleinbock, N., 2026)

For any decreasing positive function g, the set

$$\widehat{\mathbf{D}}_{m,n}[g] \setminus \{\textit{trivially singular pairs}\}$$

is uncountable and dense in $\mathrm{Mat}_{m,n} \times \mathbb{R}^m$.

Theorem (Hong, Kleinbock, N., 2026)

Let $n \geq 2$. For any $\mathbf{b} \in \mathbb{R}^m$ and any decreasing function g, the set

$$\widehat{\mathbf{D}}^{\mathbf{b}}_{m,n}[g] \setminus \{\textit{matrices for which } (A, \mathbf{b}) \textit{ is trivially singular}\}$$

is uncountable and dense in $\mathrm{Mat}_{m,n}$.

Thank you for your attention!