How (not) to cut squares into triangles?

Vasiliy Neckrasov

Brandeis University

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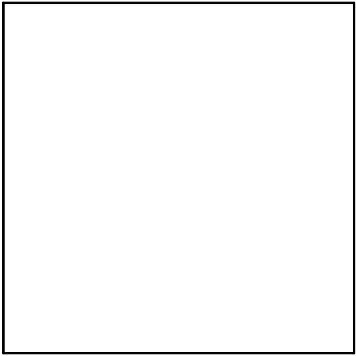
The setup

Suppose we want to dissect a square into triangles of equal area. How many triangles can we have?



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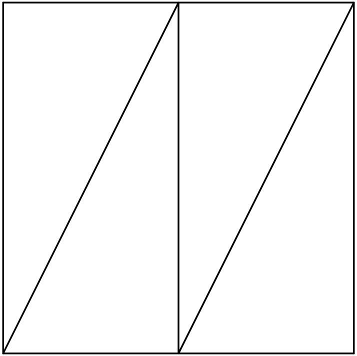

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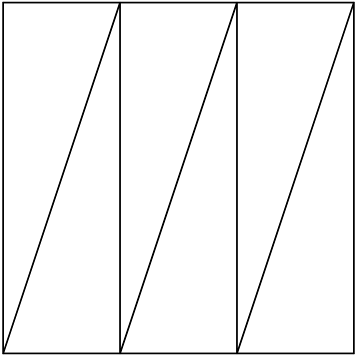

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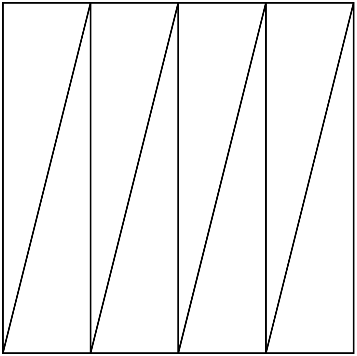

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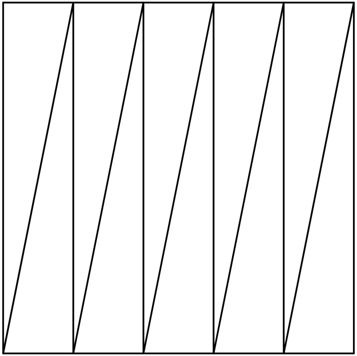

Odd number of triangles

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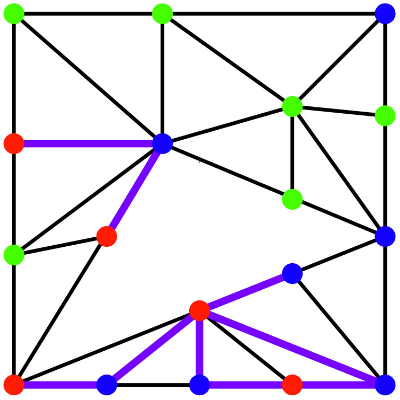
For example, we can somehow dissect a square into triangles – not controlling neither the number, nor the area – and them paint their vertices!

We paint vertices in 3 colors: blue, green and red. We call a triangle rainbow if its vertices have 3 different colors.

Lemma (Sperner)

#of rainbow triangles $\equiv$ *#RB-edges on the boundary* (*mod* 2).



First part of the story: combinatorics

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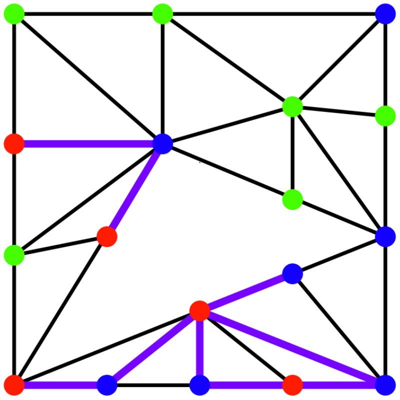

Proof. Put a dot on each side of each RB-segment and count the number N of dots inside the square twice.

- Each rainbow triangle contains 1 dot inside. Any other triangle has either 0 or 2 RB-edges, so contains 0 or 2 dots inside. Therefore,

$$N \equiv \#\text{of rainbow triangles} \pmod 2.$$

- On the other hand, each RB edge **not** on the boundary contributes 2 dots. Each RB edge on the boundary contributes 1 dot. Therefore,

$$N \equiv \#\text{of RB-edges} \pmod 2.$$

$\square$

Second part of the story: valuations

The second surprising ingredient in our construction is valuations – a way to measure the size of a rational (or real) number.

A function $\nu : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is called valuation if the following three properties hold:

1. $\nu(x) = 0$ if and only if $x = 0$;
2. $\nu(xy) = \nu(x)\nu(y)$ (**multiplicativity**)
3. $\nu(x + y) \leq \nu(x) + \nu(y)$ (**triangle inequality**)

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Example

$\nu(x) = |x|$ – absolute value of x.

Second part of the story: valuations

A more nontrivial construction is a p-**adic valuation** $|\cdot|_p$:

Let p be a prime number. Any nonzero rational number r has a **unique** representation of the form

$$r = p^k \frac{a}{b}, \qquad \text{where } a \text{ and } b \text{ are coprime to } p.$$

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Let us check that $|\cdot|_p$ is a valuation:

1. If $r \neq 0$, then $|r|_p = p^{-k} \neq 0$;
2. $|p^k \frac{a}{b} \cdot p^s \frac{c}{d}|_p = p^{-k-s} = p^{-k} p^{-s} = |p^k \frac{a}{b}|_p \cdot |p^s \frac{c}{d}|_p$;

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3. Let $r_1 = p^k \frac{a}{b}, r_2 = p^s \frac{c}{d}$ and suppose $k \geq s$. Then,

$$r_1 + r_2 = p^k \frac{a}{b} + p^s \frac{c}{d} = p^s \frac{adp^{k-s} + bc}{bd}.$$

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- If $k > s$: in this case $adp^{k-s} + bc$ is not divisible by p and so

$$|r_1 + r_2|_p = p^{-s} = max(|r_1|_p, |r_2|_p) \leq |r_1|_p + |r_2|_p;$$

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- If $k = s$: in this case $adp^{k-s} + bc$ can be divisible by p; for example, let it be divisible by p^l. Then

$$|r_1 + r_2|_p = p^{-s-l} \leq p^{-s} = max(|r_1|_p, |r_2|_p) \leq |r_1|_p + |r_2|_p.$$

In both cases the triangle inequality holds, but moreover...

We have shown a stronger property

3. * $|r_1 + r_2|_p \leq \max(|r_1|_p, |r_2|_p)$, and if $|r_1|_p \neq |r_2|_p$, then equality holds.

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A valuation which satisfies this stronger third property is called non-Archimedean. In other words, ν is a non-Archimedean valuation if

1. $\nu(x) = 0$ if and only if $x = 0$;
2. $\nu(xy) = \nu(x)\nu(y)$ (**multiplicativity**)
3. * $\nu(x + y) \leq \max(\nu(x), \nu(y))$ (**non-Archimedean property**)

It turns out that in general the property

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Indeed: suppose $\nu(x) > \nu(y)$. Then,

$$\nu(x+y) \leq \nu(x) = \nu(x+y-y) \leq \max(\nu(x+y), \nu(y)),$$

and since the inequality $\nu(x) \leq \nu(y)$ cannot hold,

$$\nu(x+y) \leq \nu(x) \leq \nu(x+y).$$

Properties of valuations.

- $$\nu(1) = \nu(1 \cdot 1) = \nu(1) \cdot \nu(1),$$

  and since $\nu(1) \neq 0$, we conclude that $\nu(1) = 1$.

- $$\nu(1) = \nu((-1) \cdot (-1)) = \nu(-1) \cdot \nu(-1),$$

  and since $\nu(-1) \neq 0$, we conclude that $\nu(-1) = 1$.

- Since

  $$1 = \nu(1) = \nu(x \cdot x^{-1}) = \nu(x) \cdot \nu(x^{-1}),$$

  we conclude that $\nu(x^{-1}) = \nu(x)^{-1}$.

An interesting fact we do not need:
The function ν defined by

$$\nu(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{otherwise} \end{cases}$$

is a real valuation of $\mathbb{Q}$. All the other valuations have one of two forms:

Theorem (Ostrowski)

Any nontrivial real valuation on $\mathbb{Q}$ is equivalent (up to raising to a positive power) to:

- *Absolute value $|\cdot|$ – if it is Archimedean; and*
- *One of p-adic valuations $|\cdot|_p$ – if it is non-Archimedean.*

To apply valuations to our problem, we need them to be defined on the whole $\mathbb{R}$ rather than $\mathbb{Q}$ – real valuations of real numbers. It turns out that valuations of rationals are quite useful as a tool to cook up valuations of reals:

Theorem (Chevalley)

Let $\nu : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ be a valuation. It can be extended to the set of real numbers; that is, exists a valuation $\mu : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$\mu(x) = \nu(x) \qquad \text{for any } x \in \mathbb{Q}.$$

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For convenience, we will use the same symbol for valuation and its extension. From now on, $|\cdot|_p$ denotes a p-adic real valuation of **real** numbers.

A bit of geometry (or algebra?)

We need one more ingredient:

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three points on the plane. The area of the triangle with vertices at these points is given by

$$\frac{1}{2}\left|\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|.$$

Theorem (P. Monsky, 1970)

A square cannot be dissected into an odd number of triangles with equal areas.

This problem was brought up in the 1960's and solved by Brandeis Math department professor Paul Monsky in 1970. It is remarkable that no other proofs of this statement (and some similar statements) are known to this day!

Clearly, we can always assume that our square is a unit square with vertices $(0,0),(1,0),(0,1)$ and $(1,1)$. We will now introduce a coloring of the whole xy-plane based on *which of the 2-adic valuations is the largest*: of x, of y or of 1.

We are painting the point:

1. **blue** if $|x|_2 \geq |y|_2$ and $|x|_2 \geq 1$ ("x is the largest");
2. **green** if $|y|_2 > |x|_2$ and $|y|_2 \geq 1$ ("y is the largest");
3. **red** if $|x|_2 < 1$ and $|y|_2 < 1$ ("1 is the largest").

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Now let us suppose that a square is dissected into triangles in some way. We can assign colors to all the vertices of these triangles using coloring we introduced.

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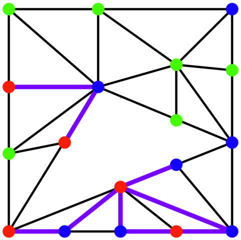

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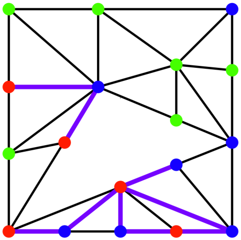
By construction:

- The point $(0,0)$ is red and the point $(1,0)$ is blue;
- Side $y = 0$ only contains red and blue points. Side $x = 0$ does not contain blue points, and sides $x = 1$ and $y = 1$ do not contain red points.



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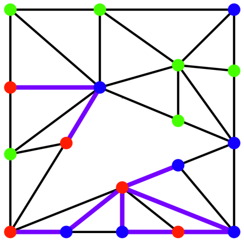
Thus, the number of RB segments on the boundary is odd. Indeed: they can only show up on the side $y = 0$, and the number of switches from red to blue going left to right has to be odd.



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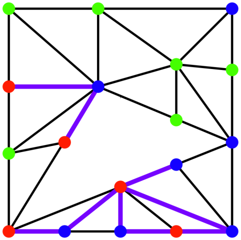
By Sperner's lemma, the number of rainbow triangles is also odd.



Lemma

The area of a rainbow triangle cannot be equal to $\frac{1}{n}$ for an odd n.

Suppose our rainbow triangle has vertices:

- (x_b, y_b) – blue;
- (x_g, y_g) – green;
- (x_r, y_r) – red.

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Let Δ be the area of this triangle. We want to show that

$$|2\Delta|_2 = \left|\det \begin{pmatrix} x_b & y_b & 1 \\ x_g & y_g & 1 \\ x_r & y_r & 1 \end{pmatrix}\right|_2 \geq 1$$

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Indeed: by construction of coloring, the diagonal elements x_b, y_g and 1 have *the largest* 2-adic valuations in their respective rows. Moreover:

$$|x_g|_2 < |y_g|_2 \quad \text{and} \quad |x_r|_2 < 1, \; |y_r|_2 < 1.$$

Note that **all the terms** in determinant calculation except for the diagonal term $x_b y_g \cdot 1$ have one of x_g, x_r or y_r as a factor.

Lemma

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So

$$|2\Delta|_2 = \left|\det\begin{pmatrix} x_b & y_b & 1 \\ x_g & y_g & 1 \\ x_r & y_r & 1 \end{pmatrix}\right|_2 = |x_b y_g|_2 = |x_b|_2 |y_g|_2 \geq 1$$

(we know that $|x_b|_2 \geq 1$ and $|y_g|_2 \geq 1$).

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(we know that $|x_b|_2 \geq 1$ and $|y_g|_2 \geq 1$).

But if we assume that $\Delta = \frac{1}{n}$ for an odd n, then $|2\Delta|_2 = \left|\frac{2}{n}\right|_2 = \frac{1}{2}\left|\frac{1}{n}\right|_2 = \frac{1}{2} < 1.$ This proves the lemma.

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- Suppose a (unit) square is dissected into n triangles with equal area. The area of each triangle is then equal to $\frac{1}{n}$
- According to the First Lemma: at least one of the triangles is a rainbow triangle. So we should have a rainbow triangle of area $\frac{1}{n}$.
- This is impossible by the Second Lemma.

Bonus: more counterintuitive statements

1. We partition the n-dimensional cube and partition into simplices. The number of simplices must be a multiple of $n!$.
2. We partition regular n-gons for $n > 4$. The number of triangles is divisible by n.
3. For a centrally symmetric polygon, the answer is the same as for the square. This was conjectured by Stein and proven by Monsky in 1990.
4. Take a polygon and try to dissect it into triangles of equal areas. There are some polygons for which this can't be done. An example is the trapezoid with vertices $(0, 0), (0, 1), (1, 0)$, and $(a, 1)$, where a is not algebraic.

Thank you for your attention!