Geometry and dynamics of continued fractions

Vasiliy Neckrasov

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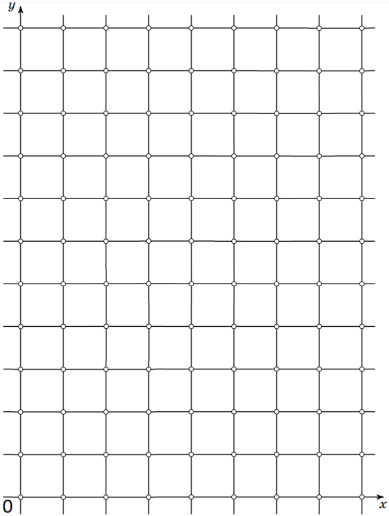
11/10/2024

Let α be a real number. Consider the following picture:



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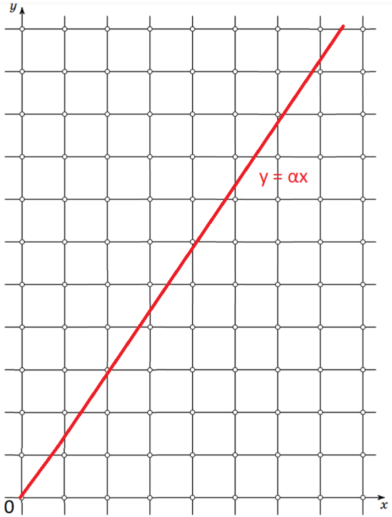
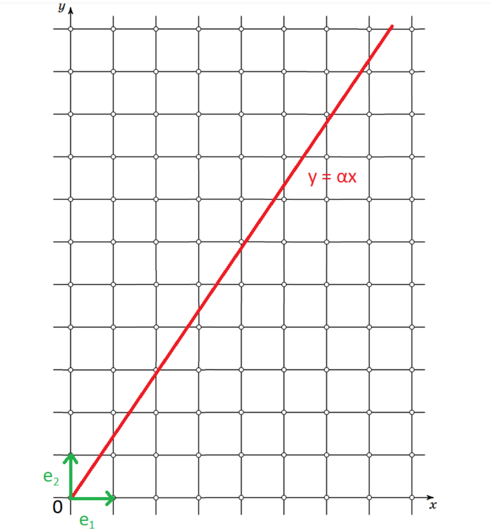
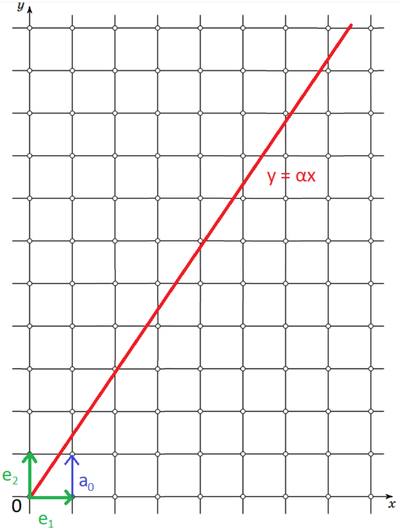


What is a continued fraction?

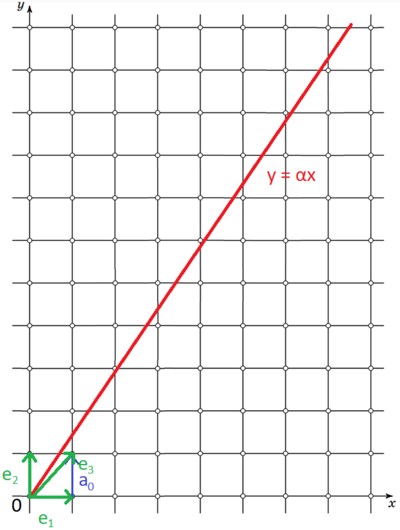
Let's introduce the vectors $\boldsymbol{e}_1 = (1, 0)$ and $\boldsymbol{e}_2 = (0, 1)$.



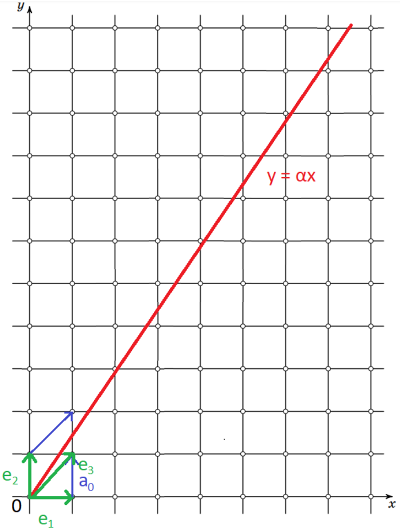
a_0 is the biggest integer, such that $\boldsymbol{e}_1 + a_0\boldsymbol{e}_2$ is under $y = \alpha x$.



We define $\boldsymbol{e}_3 = \boldsymbol{e}_1 + a_0\boldsymbol{e}_2$:



a_1 is the biggest integer, such that $\boldsymbol{e}_2 + a_1 \boldsymbol{e}_3$ is above $y = \alpha x$.



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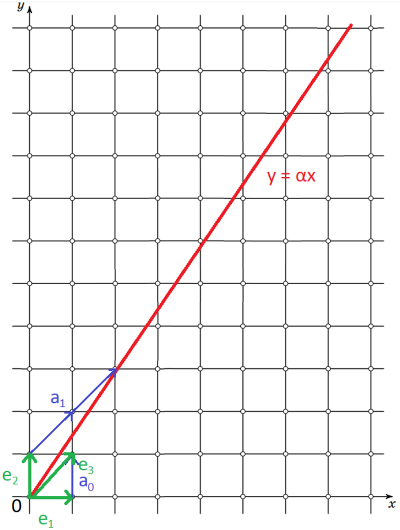

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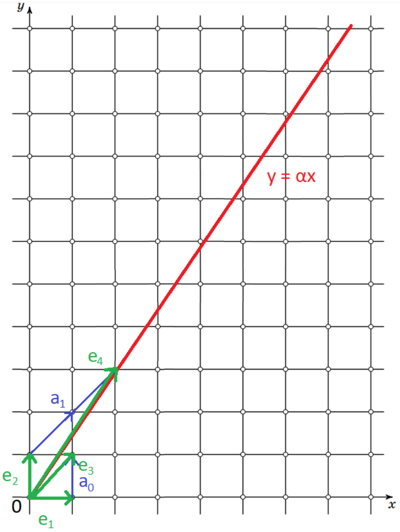
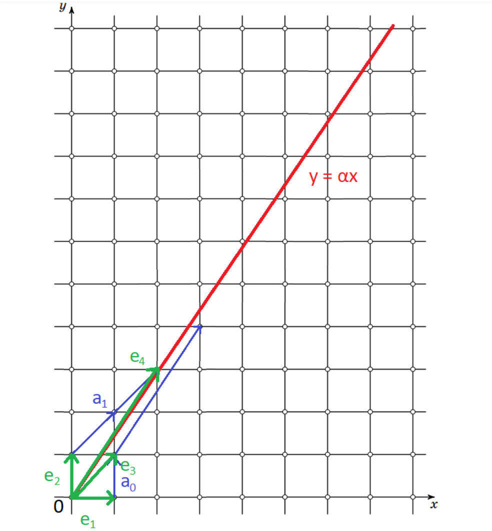


a_n is such that $\boldsymbol{e}_{n+1}$ and $a_{n+1} + a_n\boldsymbol{e}_{n+2}$ belong to the same halfplane.



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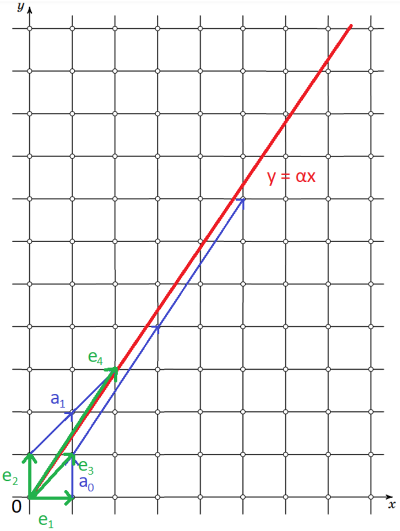

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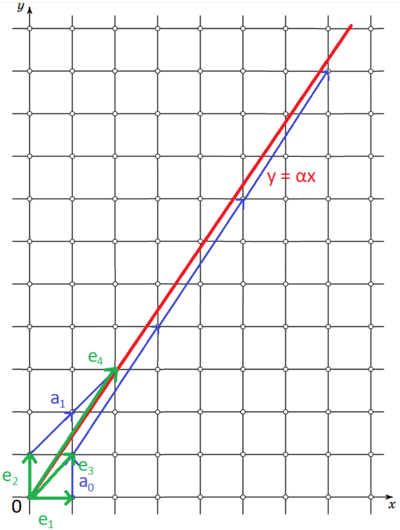

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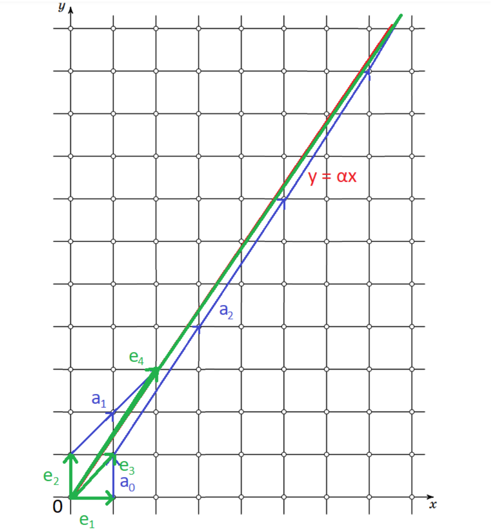

Definition

We call the elements of the sequence $a_0, a_1, a_2, \ldots$ the *partial quotients* of the continued fraction expansion of α. We write

$$\alpha = [a_0; a_1, a_2, \ldots, a_n, \ldots].$$

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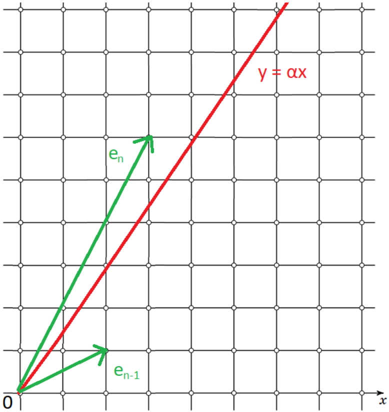
Let's also denote $\boldsymbol{e}_n := (q_n, p_n)$. Easy to see that

$$\begin{cases} q_{n+3} = a_n q_{n+2} + q_{n+1}; \\ p_{n+3} = a_n p_{n+2} + p_{n+1} \end{cases} \qquad \text{and} \qquad q_1 = p_2 = 1; \;\; q_2 = p_1 = 0.$$

The vectors $\boldsymbol{e}_n$ approximate the line $y = \alpha x$ better and better:



In particular: if $n \geq 1$, then $a_1 \geq 1$.

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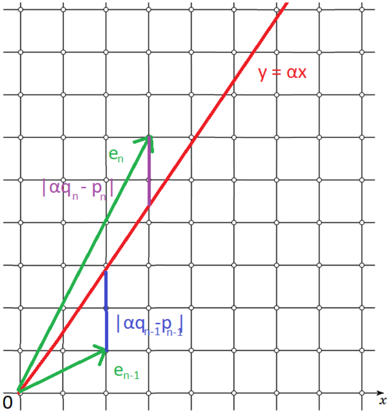

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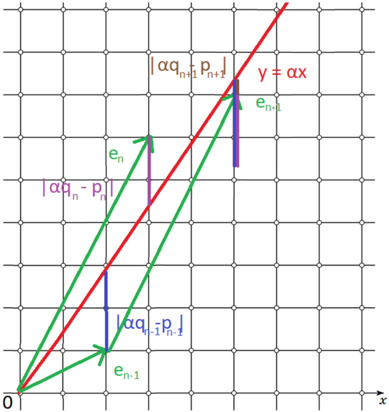

In particular: if $n \geq 1$, then $a_n \geq 1$.

It's easy to see that every real α gives us a unique continued fraction

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Let's now show that any sequence

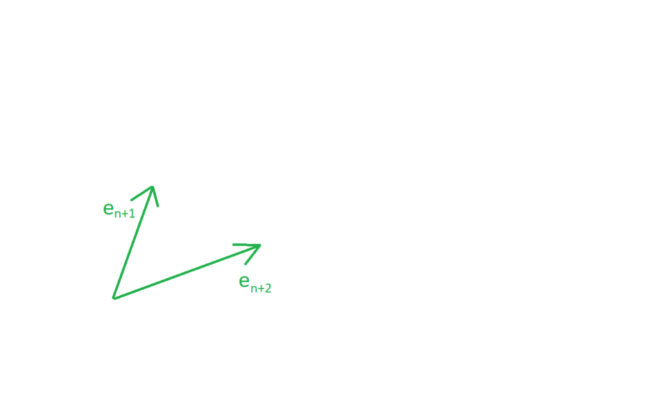
$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

with $a_n \geq 1$ for $n \geq 1$ encodes a unique α.

Lemma

A parallelogram with sides $\mathbf{e}_{n+1}$ and $\mathbf{e}_{n+2}$ has area 1 and does not contain integer points except the vertices.

Proof.



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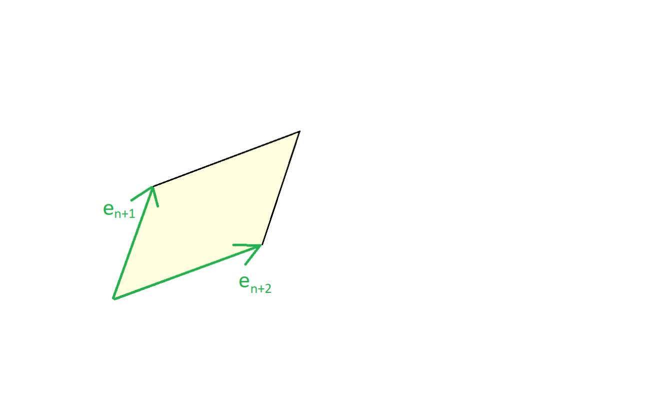

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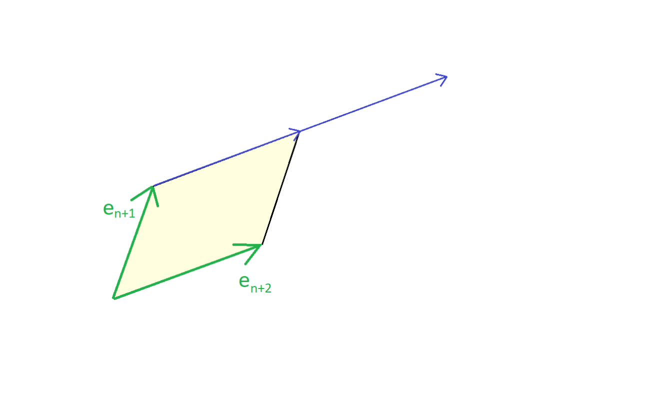

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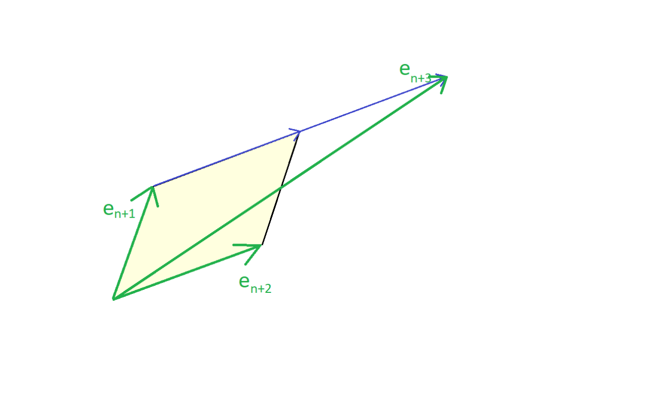

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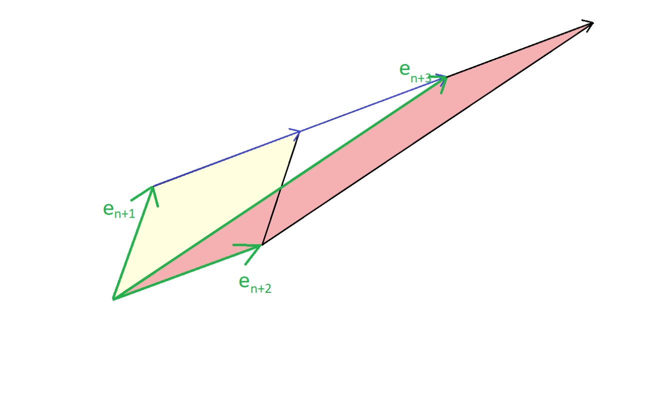

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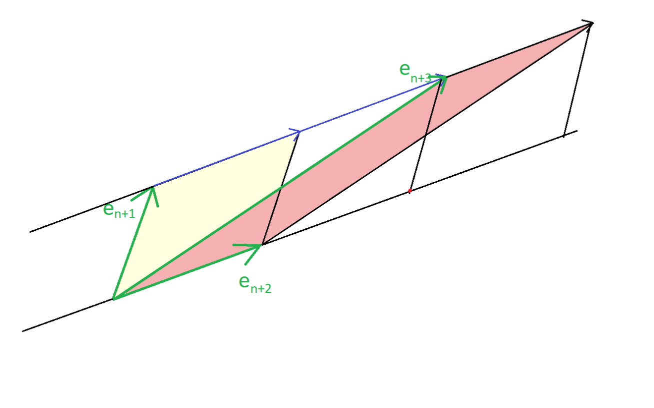

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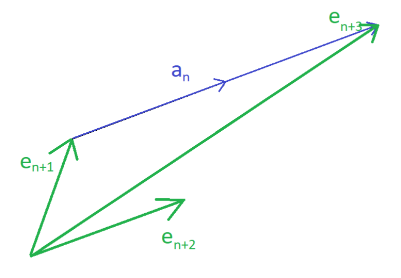
(Induction)



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First, $q_1 = 1$; $q_2 = 0$; $q_3 = q_1 + a_0 q_2 = 1$; $q_4 = q_2 + a_1 q_3 = a_1 \geq 1$.
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This proves the base case $k = 3$.
Assuming that $q_{k-1} \geq 2^{\frac{k-5}{2}}$, we get that

$$q_{k+1} = a_{k-2} q_k + q_{k-1} \geq q_k + q_{k-1} \geq 2q_{k-1} \geq 2^{\frac{k-5}{2}}.$$

Clearly, $|\boldsymbol{e}_k| \geq q_k$. $\square$

Now let φ_k be the angle between $\boldsymbol{e}_k$ and $\boldsymbol{e}_{k+1}$. The following is a corollary from our previous lemmata:

Lemma

$\varphi_k \to 0$.

Proof.

Let's notice that the area of the parallelogram generated by $\boldsymbol{e}_k$ and $\boldsymbol{e}_{k+1}$ is given by $|\boldsymbol{e}_k||\boldsymbol{e}_{k+1}|\sin\varphi_k$ and so

$$|\boldsymbol{e}_k||\boldsymbol{e}_{k+1}|\sin\varphi_k = 1;$$

$$\sin\varphi_k \le \frac{1}{|\boldsymbol{e}_k||\boldsymbol{e}_{k+1}|} \le 2^{4-k} \to 0.$$

□

Every sequence encodes a number

Theorem

For any sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of integers with an additional condition $a_n \geq 1$ for $n \geq 1$, there exists a unique real number α such that

$$\alpha = [a_0; a_1, a_2, \ldots, a_n, \ldots]$$

Proof.

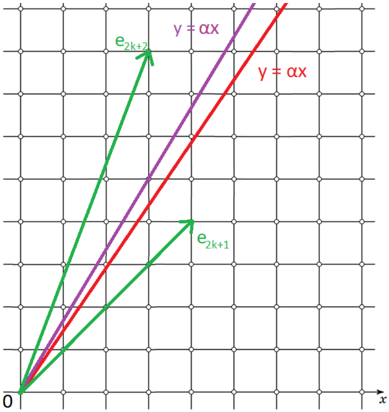
There exists such an α that all the e_{2k+1} are below $y = \alpha x$ and all the e_{2k} are above: any

$$\sup_k \{\frac{p_{2k+1}}{q_{2k+1}}\} \leq \alpha \leq \inf_k \{\frac{p_{2k}}{q_{2k}}\}$$

works. $\square$

For uniqueness: if there exists two such numbers α_1 and α_2, then the angle φ between $y = \alpha_1 x$ and $y = \alpha_2 x$ should satisfy

$$\varphi \leq \varphi_k \to 0.$$



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It turns out that these a_n are the same as the partial quotients from our geometric algorithm, and

$$\alpha = [a_0; a_1, a_2, \dots, a_n, \dots]$$

Let $T : (0,1) \setminus \mathbb{Q} \to (0,1) \setminus \mathbb{Q}$ be a map defined by

$$T(x) = \left\{ \frac{1}{x} \right\}.$$

It's called the *Gauss map*, or *continued fractions map*.

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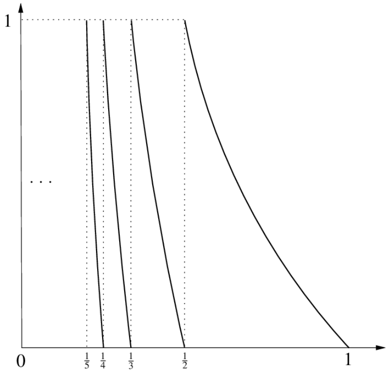
Let's draw the graph of T:



By $T^n(x)$ we mean $T(T(\ldots T(x)\ldots))$ where T is taken n times.
Let's notice that if

$$\alpha = [0; a_1, a_2, \ldots, a_n, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{\ldots + \cfrac{1}{a_n + \ldots}}},$$

then

$$T(\alpha) = \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_{n+1} + \ldots}}} = [0; a_2, a_3, \ldots, a_{n+1}, \ldots],$$

which means that T is just a shift on the space of ∞-tuples.

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which means that T is just a shift on the space of ∞-tuples.

In particular, $a_n = \lfloor \frac{1}{T^{n-1}(\alpha)} \rfloor$.

We want to introduce an unusual measure μ on the interval $(0, 1)$: for any (good enough — which can mean, for example, Borel) set A,

$$\mu(A) = \frac{1}{\ln 2} \int_A \frac{dx}{1 + x}.$$

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Now the length of an interval (a, b) is thought of as being equal to

$$\frac{1}{\ln 2} \left(\ln(1+b) - \ln(1+a) \right)$$

instead of $b - a$.

This μ is called *the Gauss measure*.

Definition

A measure μ is called T-invariant if for any (good/Borel) set $A \subseteq (0,1)$,

$$\mu(T^{-1}A) = \mu(A).$$

Here, by $T^{-1}(A)$ we mean the set of all the points x such that $T(x) \in A$.

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Proposition

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Proof.

By the properties of Borel sets, it's enough to check T-invariance only for intervals of the form $(0, s)$. This is not hard to do by hand. $\square$

Definition

Let X be a measure space (a space with some measure), and $T: X \to X$ be a measure preserving transformation. T is called *ergodic* if for any Borel $B \subset X$:

$$T^{-1}B = B \qquad \Longleftrightarrow \qquad \mu(B) = 0 \text{ or } \mu(B) = \mu(X).$$

Let S be a circle of length 1, and let $R_\alpha : \ S \to S$ be a counterclockwise rotation by $2\pi\alpha$ radians.

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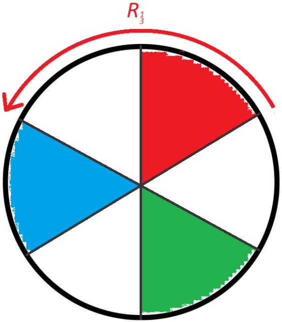
When α is rational, R_α is not ergodic:



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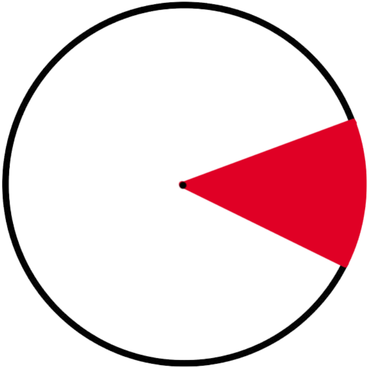

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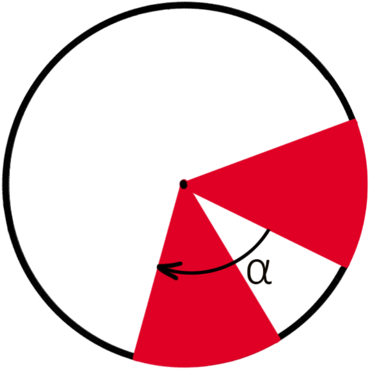

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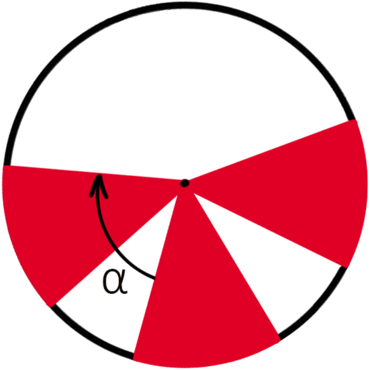

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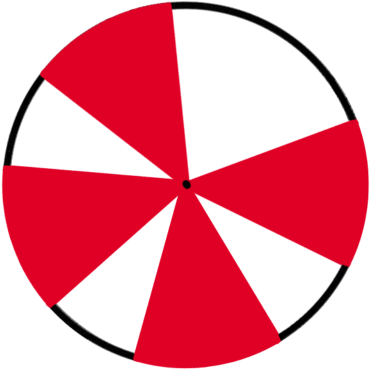

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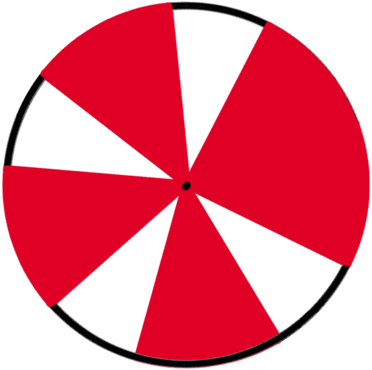

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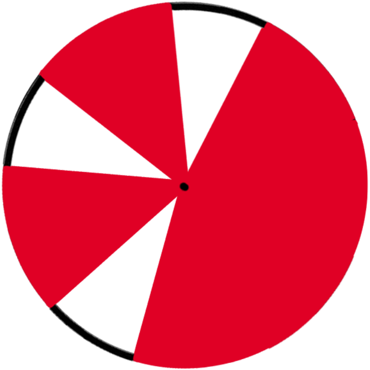

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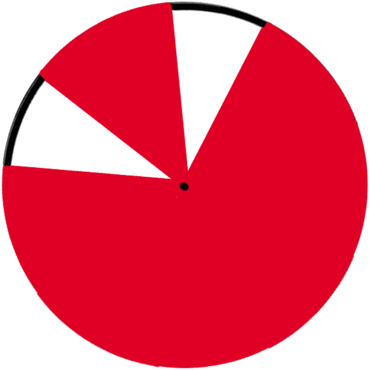

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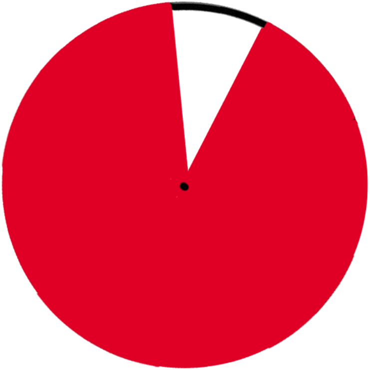

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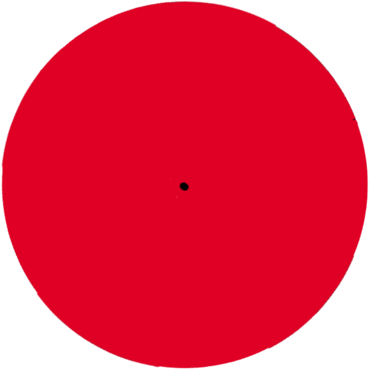

Theorem

Let f be an 1-integrable function on some measure space X with measure μ, and let $T : X \to X$ be an ergodic transformation. Then, for almost all $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int_X f d\mu.$$

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By almost all, we mean precisely the following thing: the set of x which do not satisfy your condition is very small — it has measure zero. It means that we can cover this set by a collection of open intervals of arbitrarily small total length.

Proposition

The Gauss map T is ergodic.

Theorem

(Khintchine) For almost any real number $\alpha = [a_0; a_1, a_2, \ldots, a_n, \ldots]$,

$$\lim_{n\to\infty} (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n}} = K_0,$$

where $K_0 = 2.68545\,20010\,65306\,44530\ldots$.

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Proof.

Let's define $f(x)$ by $f(x) = \ln(a)$ if $x \in \left(\frac{1}{a+1}; \frac{1}{a}\right]$. This function is 1-integrable on $(0, 1)$, and

$$\frac{1}{n}\sum_{j=1}^{n} \ln a_j = \frac{1}{n}\sum_{j=0}^{n-1} f(T^j\alpha) \to \int_0^1 f(x)d\mu = K_0$$

for almost all α by Birkhoff ergodic theorem. $\square$

Thank you for your attention!